3 Boolean Algebra Cheatsheet

3.1 Definitions

* **Boolean function** is a function of the form $f : \mathbb{B}^n \to \mathbb{B}$, where $n \geq 0$ is the *arity* of the function and $\mathbb{B} = \{0, 1\} = \{\bot, \top\} = \{\mathrm{F}, \mathrm{T}\}$ is a Boolean domain.
* There are multiple ways to represent a Boolean function (all examples represent the same function):
  1. Truth table, *e.g.*, $f = (1010)$, where LSB corresponds to $\mathbb{1}$, MSB to $\mathbb{0}$. Least/Most Significant Bit
  2. Analytically (as a sentence of propositional logic), *e.g.*, $f(A, B) = \neg B$. Propositional logic
  3. Sum of minterms, *e.g.*, $f = \sum m(0, 2) = m_0 + m_2$. Minterms
  4. Product of maxterms, *e.g.*, $f = \prod M(1, 3) = M_1 \cdot M_3$. Maxterms
  5. Boolean function number, *e.g.*, $f^{(2)}_{10}$ is the 10-th 2-ary function.
     Note that Wolfram's "Boolean operator number" is a slightly different term, which uses the reversed truth table. 10-th Boolean function $f^{(2)}_{10}$ with the truth table (1010) can be obtained via the query "5th Boolean function of 2 variables" (note: not 10th!) in WolframAlpha, since $\texttt{rev}(1010_2) = 0101_2 = 5_{10}$.

3.2 Normal Forms

* *Disjunctive* forms:
  - **Cube** is a conjunction of literals: $\mathcal{T} = \bigwedge_i \mathcal{L}_i$.
  - Formula is in **disjunctive normal form (DNF)** if it is a disjunction of terms: $\text{DNF} = \bigvee_i \mathcal{T}_i$.
  - **Minterm** is conjunction of literals, where *each* variable appears *once*, *e.g.*, $m_6 = (A \land B \land \neg C)$.
  - Formula is in **canonical DNF (CDNF)** if it is a disjunction of minterms: $\text{CDNF} = \bigvee_i m_i$.
* *Conjunctive* forms:
  - **Clause** is a disjunction of literals: $C = \bigvee_i \mathcal{L}_i$
  - Formula is in **conjunctive normal form (CNF)** if it is a conjunction of clauses: $\text{CNF} = \bigwedge_i C_i$.
  - **Maxterm** is disjunction of literals, where *each* variable appears *once*, *e.g.*, $M_6 = (\neg A \lor \neg B \lor C)$.
  - Formula is in **canonical CNF (CCNF)** if it is a conjunction of maxterms: $\text{CCNF} = \bigwedge_i M_i$.
* Some other normal forms:
  - Formula is in **negation normal form (NNF)** if the negation operator ($\neg$) is only applied to variables and the only other allowed Boolean operators are conjunction ($\land$) and disjunction ($\lor$).
  - Formula f is in **Blake canonical form (BCF)** if it is a disjunction of *all* the *prime implicants* of f.
  - Formula is in **prenex normal form (PNF)** if it consists of *prefix* — quantifiers and bound variables, and *matrix* — quatifier-free part.
  - Formula is in **Skolem normal form (SNF)** if it is in prenex normal form with only universal first-order quantifiers.
  - **Zhegalkin polynomial** is a formula in the following form (**algebraic normal form (ANF)**):
    - $f(X_1, \dots, X_n) = a_0 \oplus \bigoplus_{\substack{1 \leq i_1 \leq \dots \leq i_k \leq n \\ 1 \leq k \leq n}} (a_{i_1,\dots,i_k} \land X_{i_1} \land \dots \land X_{i_k}),\ \text{where } a_0, a_{i_1,\dots,i_k} \in \mathbb{B}$
    - $f(x_1, \dots, x_n) = a_0 \oplus (a_1 x_1 \oplus \dots \oplus a_n x_n) \oplus (a_{1,2} x_1 x_2 \oplus \dots \oplus a_{n-1,n} x_{n-1} x_n) \oplus \dots \oplus a_{1,\dots,n} x_1 \dots x_n$

3.3 Conversion to CNF/DNF

In order to convert *arbitrary* (*i.e. any*) Boolean formula to *equivalent* CNF/DNF:

1. Eliminate equivalences, implications and other "non-standard" operations (*i.e.* rewrite using only $\{\land, \lor, \neg\}$):
   $\mathcal{A} \leftrightarrow \mathcal{B} \ \rightsquigarrow\ (\mathcal{A} \to \mathcal{B}) \land (\mathcal{B} \to \mathcal{A})$
   $\mathcal{A} \to \mathcal{B} \ \rightsquigarrow\ \neg\mathcal{A} \lor \mathcal{B}$
2. Push negation downwards:
   $\neg(\mathcal{A} \lor \mathcal{B}) \ \rightsquigarrow\ \neg\mathcal{A} \land \neg\mathcal{B}$
   $\neg(\mathcal{A} \land \mathcal{B}) \ \rightsquigarrow\ \neg\mathcal{A} \lor \neg\mathcal{B}$
3. Eliminate double negation:
   $\neg\neg\mathcal{A} \ \rightsquigarrow\ \mathcal{A}$
   Note that after the recursive application of 1–3 the formula is in NNF.
4. Push disjunction (for CNF) / conjunction (for DNF) downward:
   $(\mathcal{A} \land \mathcal{B}) \lor C \ \rightsquigarrow_{\text{CNF}}\ (\mathcal{A} \lor C) \land (\mathcal{B} \lor C)$
   $(\mathcal{A} \lor \mathcal{B}) \land C \ \rightsquigarrow_{\text{DNF}}\ (\mathcal{A} \land C) \lor (\mathcal{B} \land C)$
5. Eliminate $\top$ and $\bot$:
   $\mathcal{A} \land \top \rightsquigarrow \mathcal{A}$ $\qquad$ $\mathcal{A} \land \bot \rightsquigarrow \bot$
   $\mathcal{A} \lor \top \rightsquigarrow \top$ $\qquad$ $\mathcal{A} \lor \bot \rightsquigarrow \mathcal{A}$
   $\neg\top \rightsquigarrow \bot$ $\qquad$ $\neg\bot \rightsquigarrow \top$

3.4 Functional Completeness

* A set S is called **closed** under some operation "$\bullet$" if the result of the operation applied to any elements in the set is also contained in this set, *i.e.* $\forall x, y \in S : (x \bullet y) \in S$. Closed set
* The **closure** S^* of a set S is the minimal *closed* superset of S. Closure
* A set of Boolean functions F is called **functionally complete** if it can be used to express all possible Boolean functions. Formally, $F^* = \mathbb{F}$, where F^* is a *functional closure* of F, and $\mathbb{F} = \bigcup_{n \in \mathbb{N}} \{f : \mathbb{B}^n \to \mathbb{B}\}$.

  Post's Functional Completeness Theorem. A set of Boolean functions F is functionally complete iff it contains:
  - at least one function that does *not* preserve zero, *i.e.* $\exists f \in F : f \notin T_0$, and
  - at least one function that does *not* preserve one, *i.e.* $\exists f \in F : f \notin T_1$, and
  - at least one function that is *not* self-dual, *i.e.* $\exists f \in F : f \notin S$, and
  - at least one function that is *not* monotonic, *i.e.* $\exists f \in F : f \notin M$, and
  - at least one function that is *not* linear function, *i.e.* $\exists f \in F : f \notin L$.

* A function f is **zero-preserving** iff it is `False` on the zero-valuation ($\mathbb{0} = (0, 0, \dots, 0)$):
  $f \in T_0 \leftrightarrow f(\mathbb{0}) = 0$
* A function f is **one-preserving** iff it is `True` on the one-valuation ($\mathbb{1} = (1, 1, \dots, 1)$):
  $f \in T_1 \leftrightarrow f(\mathbb{1}) = 1$
* A function f is **self-dual** iff it is dual to itself:
  $f \in S \leftrightarrow \forall x_1, \dots, x_n \in \mathbb{B} : f(x_1, \dots, x_n) = \overline{f}(\overline{x}_1, \dots, \overline{x}_n)$.
* A function f is **monotonic** iff for every increasing valuations, the function does not decrease:
  $f \in M \leftrightarrow \forall a, b \in \mathbb{B}^n : a \preceq b \to f(a) \leq f(b)$.
  Comparison of valuations $a, b \in \mathbb{B}^n$ is defined as follows:
  $a \preceq b \leftrightarrow \bigwedge\limits_{1 \leq i \leq n} (a_i \leq b_i)$
* A function f is **linear** iff its Zhegalkin polynomial is linear (*i.e.* has a degree at most 1):
  $f \in L \leftrightarrow \deg f_{\oplus} \leq 1$