Formal Logic

Discrete Math, Fall 2025

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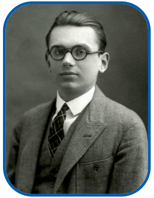
Formal Logic

“Logic is the anatomy of thought.”

— John Locke



Kurt Gödel

Alfred Tarski

T

TODO

Propositional Logic

Definition 1: Logic is the study of valid reasoning.

Definition 2: Formal logic is the study of deductively valid inferences or logical truths.

Example: *Modus ponens* inference rule:

$$\frac{\begin{array}{l} P \\ P \to Q \end{array}}{\therefore Q}$$

Definition 3 (Propositional Logic): The simplest form of logic, dealing with whole statements (*propositions*) that can be either `true` or `false`.

Also known as *sentential logic* or *zeroth-order logic*.

Syntax: The Language of Logic

Syntax concerns the formal *structure* of logical expressions — how symbols are arranged according to grammatical rules, *independent of meaning*.

Definition 4: A propositional *language* consists of:

- *Propositional variables*: $P, Q, R, \ldots$ (atomic propositions)
- *Logical connectives*: $\neg, \wedge, \vee, \rightarrow, \Longleftrightarrow$
- *Punctuation*: parentheses for grouping

Definition 5: A *well-formed formula* (WFF) in propositional logic is defined recursively:

- Every propositional variable is a WFF.
- If α is a WFF, then $\neg\alpha$ is a WFF.
- If α and β are WFFs, then $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \Longleftrightarrow \beta)$ are WFFs.
- Nothing else is a WFF.

Semantics: The Meaning of Logic

Semantics concerns the *meaning* (or *interpretation*) of logical expressions — how they relate to *truth* values and the world.

- Each propositional variable is assigned a truth value: `true` or `false`.
- More formally, an *interpretation* $\nu : V \to \mathbb{B}$ assigns truth values ($\mathbb{B} = \{0, 1\}$) to variables (atoms) V.

Definition 6: The truth value (*evaluation*) of complex formulas is determined recursively:

$$
\begin{aligned}
[\![\neg\alpha]\!]_\nu &= \texttt{true} \text{ iff } [\![\alpha]\!]_\nu = \texttt{false} \\
[\![\alpha \land \beta]\!]_\nu &= \texttt{true} \text{ iff } [\![\alpha]\!]_\nu = \texttt{true} \text{ and } [\![\beta]\!]_\nu = \texttt{true} \\
[\![\alpha \lor \beta]\!]_\nu &= \texttt{true} \text{ iff } [\![\alpha]\!]_\nu = \texttt{true} \text{ or } [\![\beta]\!]_\nu = \texttt{true} \text{ (or both)} \\
[\![\alpha \to \beta]\!]_\nu &= \texttt{false} \text{ iff } [\![\alpha]\!]_\nu = \texttt{true} \text{ and } [\![\beta]\!]_\nu = \texttt{false} \\
[\![\alpha \iff \beta]\!]_\nu &= \texttt{true} \text{ iff } [\![\alpha]\!]_\nu = [\![\beta]\!]_\nu
\end{aligned}
$$

Truth Tables

Definition 7: A *truth table* systematically lists all possible truth value assignments to propositional variables and shows the resulting truth values of complex formulas.

Example (Truth Tables for Basic Connectives):

P	$\neg P$
true	false
false	true

P	Q	$P \land Q$	$P \lor Q$
true	true	true	true
true	false	false	true
false	true	false	true
false	false	false	false

P	Q	$P \to Q$	$P \Longleftrightarrow Q$
true	true	true	true
true	false	false	false
false	true	true	false
false	false	true	true

Semantic Concepts

Definition 8: A formula φ is *satisfiable* if there exists an interpretation ν such that $[\![\varphi]\!]_\nu = \texttt{true}$.

A formula is *unsatisfiable* if no such interpretation exists.

Definition 9: A formula φ is a *tautology* (or *valid*) if $[\![\varphi]\!]_\nu = \texttt{true}$ for every interpretation ν.

Notation: $\vDash \varphi$ (read: "φ is valid").

Definition 10: A formula φ is a *contradiction* if $[\![\varphi]\!]_\nu = \texttt{false}$ for every interpretation ν.

Example:
- $P \lor \neg P$ is a tautology (Law of Excluded Middle)
- $P \land \neg P$ is a contradiction
- $P \lor Q$ is satisfiable but not a tautology

Logical Equivalence

Definition 11: Two formulas α and β are *logically equivalent*, written $\alpha \equiv \beta$, if they have the same truth value under every interpretation:

$$\alpha \equiv \beta \quad \text{iff} \quad \forall \nu.\, [\![\alpha]\!]_\nu = [\![\beta]\!]_\nu$$

Example (Important Equivalences):

De Morgan's Laws:

- $\neg(P \land Q) \equiv \neg P \lor \neg Q$
- $\neg(P \lor Q) \equiv \neg P \land \neg Q$

Double Negation:

- $\neg\neg P \equiv P$

Implication:

- $P \rightarrow Q \equiv (\neg P) \lor Q$

Associativity:

- $(P \land Q) \land R \equiv P \land (Q \land R)$
- $(P \lor Q) \lor R \equiv P \lor (Q \lor R)$

Commutativity:

- $P \land Q \equiv Q \land P$
- $P \lor Q \equiv Q \lor P$

Distributivity:

- $P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R)$

Semantic Entailment

Definition 12: A set of formulas Γ *semantically entails* a formula α, written $\Gamma \vDash \alpha$, if every interpretation that makes all formulas in Γ true also makes α true:

$$\Gamma \vDash \alpha \quad \text{iff} \quad \forall \nu. \, (\forall \beta \in \Gamma. \, [\![\beta]\!]_\nu = \texttt{true}) \to [\![\alpha]\!]_\nu = \texttt{true}$$

Example: $\{P \to Q, P\} \vDash Q$ (this captures modus ponens semantically)

Theorem 1 (Semantic Deduction Theorem): For any formulas α and β:

$$\{\alpha\} \vDash \beta \quad \text{iff} \quad \vDash \alpha \to \beta$$

Normal Forms

Definition 13: A *literal* is either a propositional variable P or its negation $\neg P$.

Definition 14: A formula is in *conjunctive normal form* (CNF) if it is a conjunction of disjunctions of literals:

$$\left(L_{1,1} \lor \ldots \lor L_{1,k_1}\right) \land \ldots \land \left(L_{n,1} \lor \ldots \lor L_{n,k_n}\right)$$

Each disjunction $\left(L_{i,1} \lor \ldots \lor L_{i,k_i}\right)$ is called a *clause*.

Definition 15: A formula is in *disjunctive normal form* (DNF) if it is a disjunction of conjunctions of literals:

$$\left(L_{1,1} \land \ldots \land L_{1,k_1}\right) \lor \ldots \lor \left(L_{n,1} \land \ldots \land L_{n,k_n}\right)$$

Normal Forms [2]

Theorem 2 (Normal Form Existence): Every propositional formula is logically equivalent to a formula in CNF and to a formula in DNF.

Example: $(P \to Q) \land R$

Converting to CNF:

1. Eliminate implications: $(\neg P \lor Q) \land R$
2. Already in CNF: $(\neg P \lor Q) \land R$

Converting to DNF:

1. Distribute: $(\neg P \land R) \lor (Q \land R)$

Boolean Satisfiability Problem (SAT)

Definition 16 (SAT Problem): Given a propositional formula φ, determine whether φ is satisfiable.

Theorem 3 (Cook-Levin Theorem): SAT is NP-complete.

From Semantics to Syntax

So far we've studied *semantics* — what formulas *mean* in terms of truth values.

Now we turn to *syntax* — how to *prove* formulas using purely symbolic manipulation, without reference to truth values.

Definition 17: A *proof system* consists of:

- *Axioms*: formulas assumed to be true
- *Inference rules*: patterns for deriving new formulas from existing ones

A *proof* of φ is a sequence of formulas ending with φ, where each formula is either an axiom or follows from previous formulas by an inference rule.

Definition 18 (Syntactic Derivability): We write $\Gamma \vdash \varphi$ (read: "Γ proves φ") if there exists a proof of φ using axioms and formulas from Γ as premises.

Natural Deduction

Definition 19 (Natural Deduction): A proof system where formulas are derived using *introduction* and *elimination* rules for each logical connective.

Proofs are typically presented in *Fitch notation* — a structured format showing the logical dependencies.

Fitch Notation

Fitch notation uses vertical lines and indentation to show proof structure:

- Vertical lines indicate scope of assumptions
- Horizontal lines separate assumptions from conclusions
- Each step is numbered and justified

Example (Fitch Proof Structure):

```
1   | P → Q          Premise
2   | P              Premise
    |____________
3   | Q              Modus Ponens 1,2
```

Inference Rules for Conjunction

Conjunction Introduction (∧I):

$$\frac{\begin{array}{l}\alpha \\ \beta\end{array}}{\alpha \land \beta}$$

If we have both α and β, we can conclude $\alpha \land \beta$.

Conjunction Elimination (∧E):

$$\frac{\alpha \land \beta}{\alpha} \qquad \frac{\alpha \land \beta}{\beta}$$

From $\alpha \land \beta$, we can conclude either α or β.

Inference Rules for Disjunction

Disjunction Introduction ($\vee$I):

$$\frac{\alpha}{\alpha \vee \beta} \qquad \frac{\beta}{\alpha \vee \beta}$$

From either α or β, we can conclude $\alpha \vee \beta$.

Disjunction Elimination ($\vee$E):

$$\frac{\begin{array}{l}\alpha \vee \beta \\ [\alpha]\ldots\gamma \\ [\beta]\ldots\gamma\end{array}}{\gamma}$$

To use $\alpha \vee \beta$, assume each disjunct and show that both lead to the same conclusion γ.

Inference Rules for Implication

Implication Introduction (→I):

$$\frac{[\alpha] \ldots \beta}{\alpha \rightarrow \beta}$$

To prove $\alpha \rightarrow \beta$, assume α and derive β.

This *discharges* the assumption α.

Implication Elimination (→E):

$$\frac{\alpha \rightarrow \beta \qquad \alpha}{\beta}$$

This is *modus ponens* — the fundamental rule of reasoning.

Inference Rules for Negation

Negation Introduction (¬I):

$$\frac{[\varphi]\ldots\bot}{\neg\varphi}$$

To prove $\neg\varphi$, assume φ and derive a contradiction $\bot$.

Negation Elimination (¬E):

$$\frac{\varphi \quad \neg\varphi}{\bot}$$

From φ and $\neg\varphi$, derive contradiction.

Definition 20: *Contradiction* ($\bot$) is special formula that represents logical inconsistency.

From $\bot$, anything can be derived (*ex falso quodlibet*).

Additional Rules

Ex Falso Quodlibet (⊥E):

$$\frac{\bot}{\varphi}$$

From contradiction, anything follows.

Double Negation Elimination:

$$\frac{\neg\neg\varphi}{\varphi}$$

(Classical logic only)

Example: Fitch Proof

Example (Proving Contrapositive): $(P \to Q) \therefore ((\neg Q) \to (\neg P))$

```
1   | P → Q                        Premise
    |________________________
2   | | ¬Q                        Assumption
    | |______________________
3   | | | P                       Assumption
4   | | | Q                       →E 1,3
5   | | | ⊥                       ¬E 2,4
    | | |____________________
6   | | ¬P                        ¬I 3-5
    | |______________________
7   | ¬Q → ¬P                     →I 2-6
    |________________________
8   | (P → Q) → (¬Q → ¬P)          →I 1-7
```

Derived Rules

Definition 21: *Derived rules* are complex inference patterns that can be proven from basic rules, used as *shortcuts* in proofs.

Example (Useful derived rules):

Modus Tollens:

$$\frac{\alpha \to \beta \qquad \neg\beta}{\neg\alpha}$$

Hypothetical Syllogism:

$$\frac{\alpha \to \beta \qquad \beta \to \gamma}{\alpha \to \gamma}$$

Proof by Contradiction (Reductio ad Absurdum):

$$\frac{[\neg\varphi]\ldots\bot}{\varphi}$$

Soundness and Completeness

Soundness of Natural Deduction

Definition 22 (Soundness): A proof system is *sound* if every syntactically derivable formula is semantically valid.

$$\text{If } \Gamma \vdash \varphi \text{ then } \Gamma \vDash \varphi$$

Theorem 4: Natural deduction for propositional logic is sound.

Proof *(sketch)*: By induction on proof structure:

Base case: Axioms and premises are semantically valid by assumption.

Inductive step: Show each inference rule preserves semantic validity:

- If premises are true under interpretation ν, then conclusion is also true under ν
- For example, for $\land$I: if $[\![\alpha]\!]_\nu = \texttt{true}$ and $[\![\beta]\!]_\nu = \texttt{true}$, then $[\![\alpha \land \beta]\!]_\nu = \texttt{true}$

The proof requires checking all inference rules systematically. □

Completeness Preview

Definition 23 (Completeness): A proof system is *complete* if every semantically valid formula is syntactically derivable.

$$\text{If } \Gamma \vDash \varphi \text{ then } \Gamma \vdash \varphi$$

Theorem 5 (Gödel): Natural deduction for propositional logic is complete.

Soundness + Completeness = syntactic derivability ($\vdash$) exactly matches semantic entailment ($\vDash$).

$$\Gamma \vdash \varphi \quad \text{iff} \quad \Gamma \vDash \varphi$$

Proof of Completeness

Proof: We prove the contrapositive: if $\Gamma \nvdash \alpha$, then $\Gamma \nvDash \alpha$.

The *strategy* is to construct a model (interpretation) that satisfies all formulas in Γ, but falsifies α.

Step 1: If $\Gamma \nvdash \alpha$, then $\Gamma \cup \{\neg\alpha\}$ is consistent (cannot derive $\bot$).

Step 2: Extend $\Gamma \cup \{\neg\alpha\}$ to a *maximal consistent set* Δ:

- Δ is consistent (cannot derive $\bot$)
- For every formula β, either $\beta \in \Delta$ or $\neg\beta \in \Delta$

Step 3: Define interpretation ν for atomic propositions P by:

$$\nu(P) = \texttt{true} \iff P \in \Delta$$

Step 4: Show by induction that for all formulas β:

$$[\![\beta]\!]_\nu = \texttt{true} \iff \beta \in \Delta$$

Step 5: Since $\neg\alpha \in \Delta$, we have $[\![\alpha]\!]_\nu = \texttt{false}$. Since $\Gamma \subseteq \Delta$, we have $[\![\gamma]\!]_\nu = \texttt{true}$ for all $\gamma \in \Gamma$.

Therefore $\Gamma \nvDash \alpha$. $\square$

The Completeness Result

Theorem 6: For any set of formulas Γ and formula φ:

$$\Gamma \vDash \varphi \quad \text{iff} \quad \Gamma \vdash \varphi$$

This establishes the *harmony* between semantics and syntax in propositional logic.

Practical implications:

- Automated theorem provers are theoretically sound.
- Truth table methods and proof methods are equivalent.
- Proof search is as hard as SAT.

Categorical Logic

"All men are mortal. Socrates is a man. Therefore, Socrates is mortal."

— Classical syllogism

A
Aristotle

S
Socrates

GB
George Boole

From Propositional to Categorical

Classical propositional logic treats statements as atomic units.

But human reasoning often involves *relationships between classes* of objects:

- "All birds can fly"
- "Some mammals are aquatic"
- "No reptiles are warm-blooded"

Traditional logic studies these patterns, providing a bridge to modern predicate logic.

Categorical Propositions

Definition 24: A *categorical proposition* is a statement that asserts or denies a relationship between two *categories* (classes) of objects.

Every categorical proposition has:

- *Subject term* (S): the category being described
- *Predicate term* (P): the category used in the description
- *Quantifier*: indicates how much of the subject is included
- *Quality*: affirmative or negative

Example: "All *politicians* are *corrupt*."

- Subject: politicians
- Predicate: corrupt people
- Quantifier: all (universal)
- Quality: affirmative

The Four Standard Forms

Definition 25: Traditional logic recognizes *four standard forms* of categorical propositions:

Form	Quantifier	Quality	Structure	Example
A	Universal	Affirmative	All S are P	"All cats are mammals"
E	Universal	Negative	No S are P	"No fish are mammals"
I	Particular	Affirmative	Some S are P	"Some birds are flightless"
O	Particular	Negative	Some S are not P	"Some animals are not vertebrates"

Examples of Categorical Propositions

A (Universal Affirmative):

- All students are hardworking
- Every theorem has a proof
- All prime numbers except 2 are odd

E (Universal Negative):

- No circles are squares
- No valid argument has false premises and true conclusion
- No even number greater than 2 is prime

I (Particular Affirmative):

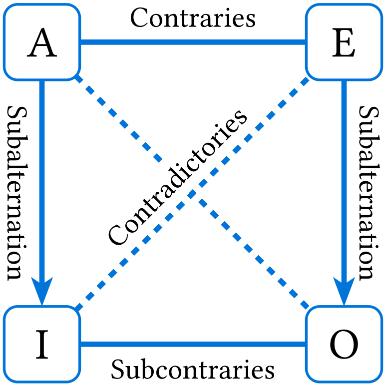
- Some politicians are honest
- Some functions are continuous
- Some equations have multiple solutions

O (Particular Negative):

- Some students are not prepared
- Some triangles are not right triangles
- Some numbers are not rational

The Square of Opposition

Definition 26: A *square of opposition* is a diagram showing the logical relationships between A, E, I, and O propositions with the same subject and predicate terms.



Logical Relationships in the Square

Definition 27 (The Four Relationships):

Contradictories (A–O, E–I):
- Cannot both be true
- Cannot both be false
- Exactly one must be true

Contraries (A–E):
- Cannot both be true
- Can both be false
- At most one is true

Subcontraries (I–O):
- Cannot both be false
- Can both be true
- At least one is true

Subalternation (A $\to$ I, E $\to$ O):
- If universal is true, particular is true
- If particular is false, universal is false

Logical Relationships in the Square [2]

Example: Given: “All roses are flowers” (A-form, `true`)

By the square of opposition:

- “No roses are flowers” (E-form) is `false` (contraries)
- “Some roses are flowers” (I-form) is `true` (subalternation)
- “Some roses are not flowers” (O-form) is `false` (contradictories)

Translation Between Traditional and Modern Logic

Definition 28: Categorical propositions can be translated into first-order logic:

Traditional	Modern Logic	Reading
All S are P	$\forall x(S(x) \to P(x))$	"For all x, if x is S then x is P"
No S are P	$\forall x(S(x) \to \neg P(x))$	"For all x, if x is S then x is not P"
Some S are P	$\exists x(S(x) \land P(x))$	"There exists x such that x is S and x is P"
Some S are not P	$\exists x(S(x) \land \neg P(x))$	"There exists x such that x is S and x is not P"

Example: "All students are hardworking" becomes: $\forall x(\text{Student}(x) \to \text{Hardworking}(x))$

"Some politicians are not honest" becomes: $\exists x(\text{Politician}(x) \land \neg\text{Honest}(x))$

The Existential Import Problem

Definition 29: A proposition "S is P" has *existential import* if it implies the existence of objects (at least one) in its subject class S.

The Problem:

Traditional logic (Aristotle) assumes all categorical propositions have existential import.

Modern logic questions this assumption.

Consider: "All unicorns are magical"
- Traditional: Implies unicorns exist (so the statement is false)
- Modern: True vacuously (if there are no unicorns, the implication holds trivially)

The Existential Import Problem [2]

Example (Impact on the Square): In modern logic with empty domains:

- A and E can both be true (if subject class is empty)
- I and O can both be false (if subject class is empty)
- Subalternation fails (A can be true while I is false)

The traditional square of opposition only works when we assume non-empty subject classes.

Syllogisms: Reasoning with Categories

Definition 30: *Categorical syllogism* is a form of reasoning with three categorical propositions:

- *Major premise*: contains the predicate of the conclusion
- *Minor premise*: contains the subject of the conclusion
- *Conclusion*: derived from the premises

Uses exactly three terms: major, minor, and middle.

Example (Classic syllogism):

All humans are mortal	(Major premise)
Socrates is human	(Minor premise)
Therefore, Socrates is mortal	(Conclusion)

Terms:

- Major term: mortal (P)
- Minor term: Socrates (S)
- Middle term: human (M)

Figures and Moods of Syllogisms

Definition 31: The *figure* of a syllogism is determined by the position of the middle term:

Figure 1	Figure 2	Figure 3	Figure 4
$\mathbf{M} - P$	$P - \mathbf{M}$	$\mathbf{M} - P$	$P - \mathbf{M}$
$S - \mathbf{M}$	$S - \mathbf{M}$	$\mathbf{M} - S$	$\mathbf{M} - S$
$S - P$	$S - P$	$S - P$	$S - P$

Definition 32: The *mood* of a syllogism is the 3-letter sequence of categorical forms (A, E, I, O) of its three propositions, in order: major premise, minor premise, conclusion.

Example (Barbara (AAA-1)):

All M are P **(A)**

Figures and Moods of Syllogisms [2]

All S are M **(A)**

All S are P **(A)**

This arguments has mood AAA in figure 1, called “Barbara” — a valid syllogistic form.

Valid Syllogistic Forms

Traditional logic identified 24 valid syllogistic forms across the four figures.

Each valid form has a traditional Latin name that encodes its mood:
- Vowels indicate the categorical forms (A, E, I, O)
- Some consonants indicate required operations for reduction

Example (Famous Valid Forms):

Figure 1	**Figure 2**	**Figure 3**	**Figure 4**
Barbara (AAA)	Cesare (EAE)	Darapti (AAI)	Bramantip (AAI)
Celarent (EAE)	Camestres (AEE)	Disamis (IAI)	Camenes (AEE)
Darii (AII)	Festino (EIO)	Datisi (AII)	Dimaris (IAI)
Ferio (EIO)	Baroco (AOO)	Felapton (EAO)	Fesapo (EAO)
		Bocardo (OAO)	Fresison (EIO)
		Ferison (EIO)	

Syllogistic Fallacies

Common syllogistic fallacies:

Fallacy of Four Terms:
Using more than three distinct terms

Example:
- All banks are financial institutions
- The river bank is muddy
- Therefore, some financial institutions are muddy

(Equivocates on “bank”)

Undistributed Middle:
Middle term not distributed in either premise

Example:
- All cats are mammals
- All dogs are mammals
- Therefore, all cats are dogs

(“Mammals” not distributed)

Syllogistic Fallacies [2]

Definition 33 (More Fallacies):

Illicit Major: Major term distributed in conclusion but not in major premise

Illicit Minor: Minor term distributed in conclusion but not in minor premise

Fallacy of Exclusive Premises: Both premises negative

Existential Fallacy: Particular conclusion from universal premises (when subject class may be empty)

Distribution of Terms

Definition 34: A term is *distributed* in a proposition if the proposition says something about *all* members of the class denoted by that term.

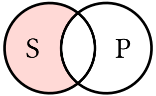
- Only *universal* propositions (A, E) distribute their *subject* term.
- Only *negative* propositions (E, O) distribute their *predicate* term.

	Distribution	
Form	Subject	Predicate
A: All S are P	✓	✗
E: No S are P	✓	✓
I: Some S are P	✗	✗
O: Some S are not P	✗	✓

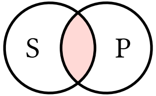
Why Distribution Matters

Example: Consider the terms in these propositions:

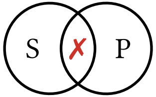
- "All cats are mammals" (*A-form*)
  - Says something about ALL cats (subject distributed)
  - Says nothing about ALL mammals (predicate not distributed)



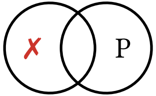
- "No reptiles are mammals" (*E-form*)
  - Says something about ALL reptiles (subject distributed)
  - Says something about ALL mammals (predicate distributed)



- "Some birds are flightless" (*I-form*)
  - Says something about SOME birds (subject not distributed)
  - Says something about SOME flightless creatures (predicate not distributed)



- "Some animals are not vertebrates" (*O-form*)
  - Says something about SOME animals (subject not distributed)
  - Says something about ALL vertebrates (predicate distributed)



Rules for Valid Syllogisms

Definition 35 (Validity Rules): A categorical syllogism is valid if and only if it satisfies all these rules:

1. **Exactly three terms** (no equivocation)
2. **Middle term distributed at least once**
3. **No term distributed in conclusion unless distributed in premise**
4. **No conclusion from two negative premises**
5. **Negative conclusion if and only if exactly one negative premise**
6. **No particular conclusion from two universal premises** (if existential import assumed)

Venn Diagrams for Categorical Logic

Definition 36 (Venn Diagram Method): Categorical propositions can be represented using Venn diagrams with two or three circles.

- Shaded regions represent empty classes
- ✗ marks represent existing individuals
- Overlap patterns show relationships between categories

Example (Venn Diagram for Syllogism): Testing Barbara (AAA-1):

- All M are P: Shade M outside P
- All S are M: Shade S outside M
- Conclusion: All S are P

The diagrams show that S must be entirely within P, validating the syllogism.

Modern Developments

Traditional categorical logic has evolved in several directions:

- **Set theory**: Categories become sets, relations become set operations
- **Formal semantics**: Precise treatment of quantification and scope
- **Knowledge representation**: Description logics in AI and semantic web
- **Natural language processing**: Computational linguistics and parsing
- **Database theory**: Query languages and constraint systems

Limitations of Traditional Logic

Traditional categorical logic has important *limitations*:

1. Only handles simple quantification (all, some, no)
2. Cannot express complex relationships (between more than two categories)
3. Limited to categorical structure (subject–predicate form)
4. Struggles with relational statements ("John is taller than Mary")
5. No systematic treatment of compound statements
6. Existential import controversies

Example (What traditional logic cannot express):

- "Every student likes some professor" (multiple quantifiers)
- "If John is happy, then Mary is happy" (conditional with individuals)
- "All numbers between 5 and 10 are prime" (complex domain restrictions)
- "Most birds can fly" (non-standard quantifiers)
- "Students who study hard usually succeed" (statistical generalizations)

The Legacy of Traditional Logic

Enduring Contributions:

- Systematic study of quantification and categorical reasoning
- Recognition of logical form vs. content
- Analysis of validity in natural language arguments
- Foundation for formal semantics and knowledge representation
- Critical thinking tools for evaluating everyday reasoning

Modern Relevance: Traditional logic remains important for understanding human reasoning patterns, developing AI systems that interact naturally with humans, and teaching critical thinking skills.

First-Order Logic

Transition to First-Order Logic

Propositional logic can only reason about *whole statements*.

To reason about *objects* and their *properties*, we need *first-order logic* (FOL).

Example (Limitations of Propositional Logic): Cannot express:

- “All humans are mortal”
- “Socrates is human”
- “Therefore, Socrates is mortal”

In propositional logic, these would be *unrelated* atomic propositions P, Q, R, without any structure connecting them.

Transition to First-Order Logic [2]

Definition 37: First-order logic extends propositional logic with:

- *Variables*: $x, y, z, \ldots$
- *Predicates*: $P(x), R(x, y), \ldots$
- *Quantifiers*: $\forall x$ (for all), $\exists x$ (there exists)
- *Functions*: $f(x), g(x, y), \ldots$
- *Constants*: $a, b, c, \ldots$

First-Order Syntax

Definition 38 (Terms): *Terms* are expressions denoting objects:

- Variables: x, y, z
- Constants: a, b, c
- Function applications: $f(t_1, \ldots, t_n)$ where t_i are terms

Definition 39 (Atomic Formulas): *Atomic formulas* are basic statements:

- Predicate applications: $P(t_1, \ldots, t_n)$ where t_i are terms
- Equality: $t_1 = t_2$ where t_1, t_2 are terms

Definition 40 (First-Order Formulas): Built recursively from atomic formulas using:

- Propositional connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\Longleftrightarrow$
- Quantifiers: $\forall x.\varphi$, $\exists x.\varphi$

First-Order Syntax [2]

Examples:

- $\forall x.\,(P(x) \to Q(x))$ — "For all x, if $P(x)$ then $Q(x)$"
- $\exists x.\,(P(x) \land \neg Q(x))$ — "There exists an x such that $P(x)$ and not $Q(x)$"
- $\forall x.\exists y.\,R(x, y)$ — "For every x, there exists a y such that $R(x, y)$"

First-Order Semantics

Definition 41: A *structure* $\mathcal{M} = \langle D, \mathcal{I} \rangle$ consists of:

- *Domain* D: non-empty set of objects
- *Interpretation function* $\mathcal{I}$:
  - Maps constants to elements of D
  - Maps n-ary predicates to n-ary relations on D
  - Maps n-ary functions to n-ary functions on D

Definition 42: A *variable assignment* $\sigma : V \to D$ maps variables to domain elements.

Definition 43 (Truth in a Structure): For structure $\mathcal{M}$ and assignment σ:

- $\mathcal{M}, \sigma \vDash P(t_1, \ldots, t_n)$ iff $\langle \mathcal{I}(t_1)^\sigma, \ldots, \mathcal{I}(t_n)^\sigma \rangle \in \mathcal{I}(P)$
- $\mathcal{M}, \sigma \vDash \forall x. \varphi$ iff $\mathcal{M}, \sigma' \vDash \varphi$ for all σ' that differ from σ at most on x
- $\mathcal{M}, \sigma \vDash \exists x. \varphi$ iff $\mathcal{M}, \sigma' \vDash \varphi$ for some σ' that differs from σ at most on x

Theories and Models

Definition 44: A *theory* T is a set of first-order formulas (axioms).

Definition 45: A structure $\mathcal{M}$ is a *model* of theory T if $\mathcal{M} \vDash \varphi$ for every formula $\varphi \in T$.

Example (Group Theory): The theory of groups has axioms:

- (Associativity) $\forall x, y, z.\, (x \cdot (y \cdot z)) = ((x \cdot y) \cdot z)$
- (Identity) $\exists e. \forall x.\, (x \cdot e = x) \land (e \cdot x = x)$
- (Inverses) $\forall x. \exists y.\, (x \cdot y = e) \land (y \cdot x = e)$

Models include $\langle \mathbb{Z}, + \rangle$, $\langle \mathbb{R} \setminus \{0\}, \cdot \rangle$, etc.

First-Order Natural Deduction

Definition 46: Additional rules for quantifiers:

Universal Introduction (∀I):

$$\frac{\varphi(a)}{\forall x.\, \varphi(x)}$$

Where a is arbitrary (fresh).

Universal Elimination (∀E):

$$\frac{\forall x.\, \varphi(x)}{\varphi(t)}$$

Existential Introduction (∃I):

$$\frac{\varphi(t)}{\exists x.\, \varphi(x)}$$

Existential Elimination (∃E):

$$\frac{\exists x.\, \varphi(x) \qquad [\varphi(a)] \ldots \psi}{\psi}$$

Where a is fresh and doesn't occur in ψ.

Interactive Theorem Provers

Modern mathematics increasingly uses *interactive theorem provers* — computer systems that assist in constructing and verifying formal proofs.

Examples (Major Systems):

Lean 4:

- Functional programming
- Dependent types
- Growing math library

Coq:

- Constructive logic
- Curry-Howard correspondence
- Machine-checked proofs

Isabelle/HOL:

- Higher-order logic
- Automated tactics
- Large formalizations

Example: Major theorems *proven* in interactive systems:

- Four Color Theorem (Coq)
- Odd Order Theorem (Coq)
- Kepler Conjecture (Isabelle/HOL)
- Liquid Tensor Experiment (Lean)

Completeness and Decidability

Theorem 7 (Gödel): First-order logic is complete: every semantically valid formula is provable.

Theorem 8 (Church): First-order logic is undecidable: there is no algorithm that determines whether an arbitrary first-order formula is valid.

The trade-off:

- Propositional logic: decidable (SAT-solvable) but has *limited expressiveness*
- First-order logic: highly expressive but *undecidable*
- Higher-order logic: even more expressive but *incomplete*

Applications and Connections

Example (Logic in Computer Science):

Verification:
- Program correctness
- Hardware verification
- Protocol analysis
- Security properties

AI and Knowledge Representation:
- Expert systems
- Automated planning
- Semantic web (RDF, OWL)
- Natural language processing

Databases:
- Query languages (SQL)
- Integrity constraints
- Deductive databases

Programming Languages:
- Type systems
- Specification languages
- Logic programming (Prolog)

Summary: The Logical Landscape

Logic	Expressiveness	Decidability	Completeness
Propositional	Basic	✓	✓
First-Order	High	✗	✓
Second-Order	Very High	✗	✗
Higher-Order	Maximum	✗	✗

Key insights:

- Syntax and semantics can be perfectly aligned (completeness)
- Expressiveness comes at the cost of decidability
- Formal logic provides foundations for mathematical reasoning and computation
- Interactive theorem provers make formal logic practically useful

Looking Forward

Next topics in advanced logic:

- Modal logic (necessity, possibility, knowledge, belief)
- Temporal logic (time, concurrency, reactive systems)
- Intuitionistic logic (constructive mathematics)
- Linear logic (resource-aware reasoning)
- Description logics (knowledge representation, semantic web)

Connections to other areas:

- Computability theory and complexity
- Category theory and type theory
- Model theory and set theory
- Philosophical logic and foundations of mathematics

TODO

- ...