Set Theory

Discrete Math, Fall 2025

Konstantin Chukharev

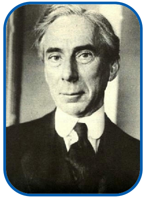
Set Theory

"A set is a Many that allows itself to be thought of as a One."

— Georg Cantor

Georg Cantor

Richard Dedekind



Bertrand Russell

Ernst Zermelo

Abraham Fraenkel

Introduction

Set theory provides a foundational language for all of mathematics. *Everything* from numbers and functions to spaces and relations can be defined using *sets*. This lecture introduces the basic objects and operations of set theory and explores their deep structural and logical consequences.

Topics include:

- Basic concepts: elements, subsets, operations
- Relations and functions as sets
- Infinite sets and cardinality
- Axiomatic foundations
- Applications in logic and computer science

Basic Notions

Definition 1: A *set* is an unordered collection of distinct objects, called *elements*.

- In *naïve* set theory, sets can contain *any* objects (including non-sets, called *urelements*).
- In modern *axiomatic* set theory, *everything is a set* (no urelements).

Example: $A = \{5, 🐨, 🐦\}$ is a set of three elements: the number 5, a koala, and a birb.

Notation: $a \in A$ means "a is *an element of* A".

Example: "$🐨 \in A$" is `true`, while "$🐧 \in A$" is `false`, denoted as "$🐧 \notin A$".

Definition 2 (Extensionality): Two sets are *equal*, denoted $A = B$, if they have the same elements, that is, iff every element of A is also in B, and vice versa. Formally, $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Example: $\{a, b, b\} = \{a, b\} = \{b, a\} = \{b, a, b\}$, all these denote the same set with elements a and b.

Set-Builder Notation

Definition 3: A set can be defined using *set-builder notation* (*set comprehension*):

$$A = \{x \mid P(x)\}$$

meaning “the set of all x such that the property $P(x)$ holds”.

Example: $A = \{x \mid x \in \mathbb{N} \land x > 5\}$ is the set of natural numbers greater than 5, that is, $A = \{6, 7, 8, \ldots\}$.

Example: $S = \{x^2 \mid x \text{ is prime}\} = \{4, 9, 25, 49, \ldots\}$ is the set of squares of prime numbers[1].

Example: $\mathbb{Q} = \{a/b \mid a \in \mathbb{Z}, b \in \mathbb{N}, b \neq 0\}$ is the set of rational numbers (fractions).

[1] **Note:** 1 *is not* a prime number.

Subsets

Definition 4: A set A is a *subset* of B, denoted $A \subseteq B$, if every element of A is also an element of B.

- Formally, $A \subseteq B \iff \forall x.\, (x \in A) \rightarrow (x \in B)$.
- If A is not a subset of B, we write $A \nsubseteq B$.
- If $A \subseteq B$ and $A \neq B$, we say A is a *proper (or strict) subset* of B, denoted $A \subset B$ or $A \subsetneq B$.
- If A is a subset of B, denoted $A \subseteq B$, then B is a *superset* of A, denoted $B \supseteq A$.

Example: Every set is a subset of itself: $A \subseteq A$.

Example: The empty set is a subset of every set: $\varnothing \subseteq A$ for any set A.

Example: The set of even numbers is a proper subset of the set of integers: $\mathbb{Z}_{\text{even}} \subset \mathbb{Z}$.

Example: $\{a, b\} \subseteq \{a, b, c\}$, but $\{a, b, x\} \nsubseteq \{a, b, c\}$.

Example: $\{0\} \in \{0, \{0\}\}$ *and* $\{0\} \subseteq \{0, \{0\}\}$, that is, $\{0\}$ is an element, and also a subset.

Power Sets

Definition 5: The *power set* of a set A, denoted 2^A or $\mathcal{P}(A)$, is the set of all subsets of A.

$$\mathcal{P}(A) = \{S \mid S \subseteq A\}$$

Example: If $A = \{a, b\}$, then $\mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$.

Example: If $A = \{1, 2, 3\}$, then $\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Example: The power set of the empty set is $\mathcal{P}(\varnothing) = \{\varnothing\}$, a *non-empty* set containing the empty set.

Theorem 1: $|\mathcal{P}(A)| = 2^{|A|}$ for any finite set A.

Proof *(combinatorial)*: For each of the n elements in the set, we can either include it in a subset or not. These n independent binary choices yield 2^n possible subsets by the multiplication principle.

$$\underbrace{2 \times 2 \times \ldots \times 2}_{n \text{ times}} = 2^n$$

□

Power Sets [2]

Proof: By *induction* on $n = |A|$, the cardinality of the set A.

Base case: If $n = 0$, then $A = \varnothing$ and $\mathcal{P}(A) = \{\varnothing\}$. Thus, $|\mathcal{P}(A)| = 1 = 2^0$.

Inductive step: Assume the formula holds for any set of size k. Let A be a set with $|A| = k + 1$. Choose an arbitrary element $a \in A$ and let $A' = A \setminus \{a\}$, so $|A'| = k$.

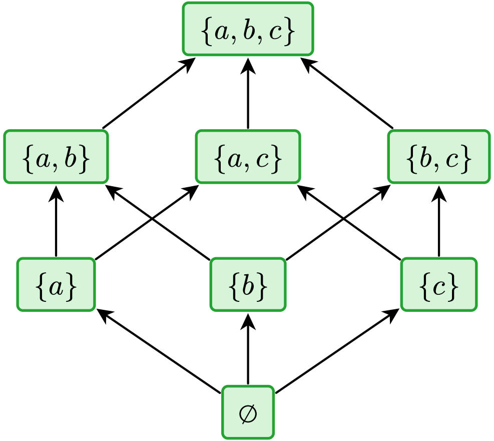
The power set $\mathcal{P}(A)$ can be partitioned into two *disjoint* collections:

1. Subsets of A that *do not* contain a. This collection is exactly $\mathcal{P}(A')$. By the inductive hypothesis, it has $|\mathcal{P}(A')| = 2^k$ elements.
2. Subsets of A that *do* contain a. Each such subset is of the form $S \cup \{a\}$ where $S \subseteq A'$. This establishes a bijection with $\mathcal{P}(A')$, so this collection also has 2^k elements.

The total number of subsets of A is the *sum* of their sizes: $|\mathcal{P}(A)| = 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$. □

Hasse Diagram of Power Set

The elements of the power set of $\{a, b, c\}$ ordered with respect to inclusion ($\subseteq$):



Some Important Sets

Example: $\mathbb{N} = \{0, 1, 2, ...\}$ is the set of natural numbers.

Example: $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ is the set of integers.

Example: $\mathbb{Q} = \{a/b \mid a \in \mathbb{Z}, b \in \mathbb{N}, b \neq 0\}$ is the set of rational numbers.

Example: $\mathbb{R} = (-\infty, +\infty)$ is the set of real numbers (the continuum).

Example: $\mathbb{B} = \{0, 1\}$ is the set of Boolean values (truth values).

Example: The set A^* of *finite strings* over an alphabet A is defined as:

$$A^* = \{\varepsilon\} \cup \{a_1 a_2 ... a_n \mid n \in \mathbb{N}, a_i \in A\} = \bigcup_{n \in \mathbb{N}} A^n$$

For example, $\mathbb{B}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 001, 010, 011, 100, 101, 110, 111, 0000, ...\}$.

Example: The set A^ω of *infinite sequences* over A.

Operations on Sets

Operation	Notation	Formal definition
Union	$A \cup B$	$\{x \mid x \in A \lor x \in B\}$
Intersection	$A \cap B$	$\{x \mid x \in A \land x \in B\}$
Difference	$A \setminus B$	$\{x \mid x \in A \land x \notin B\}$
Symmetric diff.	$A \triangle B$	$(A \setminus B) \cup (B \setminus A)$
Complement	$\overline{A}$ or A^c	$\{x \mid x \notin A\}$
Power set	2^A or $\mathcal{P}(A)$	$\{S \mid S \subseteq A\}$

Venn Diagrams and Euler Circles

Definition 6: A *Venn diagram* is a visual representation of sets and their relationships using overlapping circles or closed curves. Each circle represents a set, and overlapping regions show intersections.

Definition 7: *Euler circles* (or *Euler diagrams*) are a simpler form where circles may or may not overlap, and non-overlapping regions represent disjoint sets.

Example: For sets $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$:

- $A \cap B = \{2, 3\}$ (overlapping region)
- $A \setminus B = \{1\}$ (left-only region)
- $B \setminus A = \{4\}$ (right-only region)
- $A \cup B = \{1, 2, 3, 4\}$ (entire diagram)

Laws of Set Operations

For any sets A, B, and C:

Commutative Laws:

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$

Associative Laws:

- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Laws:

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De Morgan's Laws:

- $\overline{A \cup B} = \overline{A} \cap \overline{B}$
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Identity Laws:

- $A \cup \varnothing = A$, $A \cap U = A$ (where U is the universal set)
- $A \cap \varnothing = \varnothing$, $A \cup U = U$

Complement Laws:

- $A \cup \overline{A} = U$, $A \cap \overline{A} = \varnothing$
- $\overline{\overline{A}} = A$ (double complement)

Tuples and Ordered Pairs

Definition 8: A *tuple* is an ordered collection of elements, denoted $(a_1, a_2, \ldots, a_n)$.

A tuple of length n is called an *n-tuple*.

Example: $(42, \text{🦀}, \text{😾}, \text{🥝})$ is a 4-tuple.

Definition 9: An ordered pair $\langle a, b \rangle$ is a special 2-tuple, defined[2] as:

$$\langle a, b \rangle \overset{\text{def}}{=} \{\{a\}, \{a, b\}\}$$

Example: $\langle \text{🎃}, \text{🧙‍♀️} \rangle \neq \langle \text{🧙‍♀️}, \text{🎃} \rangle$, these are different ordered pairs.

Example: $\langle \text{🌵}, \text{🌵} \rangle \neq (\text{🌵},) \neq \text{🌵} \neq \{\text{🌵}\}$, these are all different objects: an ordered pair, a 1-tuple, an urelement, and a singleton set. Note, however, that $\langle \text{🌵}, \text{🌵} \rangle = \{\{\text{🌵}\}\}$.

[2] Kuratowski's definition is the most cited and now-accepted definition of an ordered pair. For others, see wiki.

Cartesian Product

Definition 10: The *Cartesian product* of two sets A and B, denoted $A \times B$, is defined as:

$$A \times B = \{\langle a, b \rangle \mid a \in A \text{ and } b \in B\}$$

Example: If $A = \{1, 2\}$ and $B = \{x, y, z\}$, then their product is

$$A \times B = \{\langle 1, x \rangle, \langle 1, y \rangle, \langle 1, z \rangle, \langle 2, x \rangle, \langle 2, y \rangle, \langle 2, z \rangle\}$$

Definition 11: The *n-fold Cartesian product* (also known as *Cartesian power*) of a set A is defined as:

$$A^n = \underbrace{A \times A \times \ldots \times A}_{n \text{ times}} = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A\}$$

Example: $\{a, b\}^3 = \{(a, a, a), (a, a, b), (a, b, a), (a, b, b), (b, a, a), (b, a, b), (b, b, a), (b, b, b)\}$

Example: $\{🦅\}^3 = \{(🦅, 🦅, 🦅)\}$, the singleton set containing the 3-tuple of three eagles.

Example: $A^0 = \{()\}$, the singleton set containing the empty tuple.

Geometric Interpretation of Cartesian Product

The Cartesian product $A \times B$ can be visualized as a region on the coordinate plane, where each point $\langle a, b\rangle$ represents an element of the product.

Example: If $A = [1, 4)$ and $B = (2, 4]$, then $A \times B$ represents the rectangular region: $\{(x, y) \mid 1 \leq x < 4 \text{ and } 2 < y \leq 4\}$

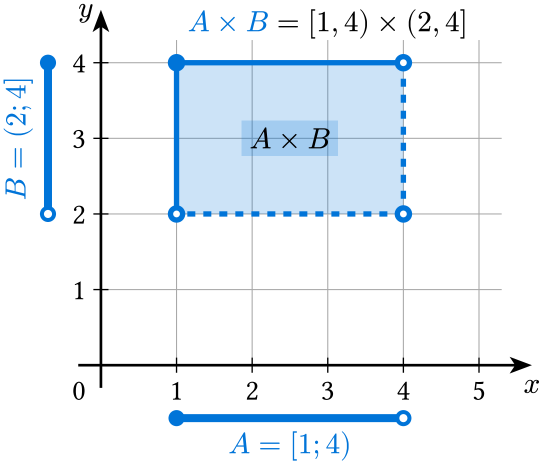
Example: For discrete sets $A = \{1, 2, 3\}$ and $B = \{1, 2\}$, the product $A \times B$ consists of 6 points: $(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)$ arranged in a grid pattern.

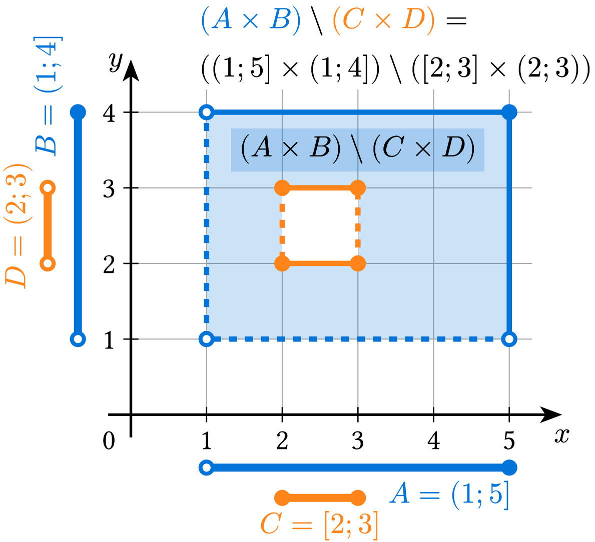
Example: The set difference $(A \times B) \setminus (C \times D)$ where:

- $A \times B = [0, 3] \times [0, 2]$ (outer rectangle)
- $C \times D = (1, 2) \times (0.5, 1.5)$ (inner rectangle to subtract)

Results in an “L-shaped” region.

Geometric Interpretation of Cartesian Product [2]

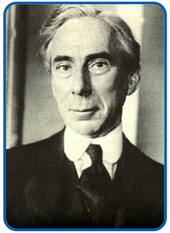



$(A \times B) \setminus (C \times D) =$
$((1; 5] \times (1; 4]) \setminus ([2; 3] \times (2; 3))$
y
x
0
1
2
3
4
5
1
2
3
4
$(A \times B) \setminus (C \times D)$
$B = (1; 4]$
$D = (2; 3)$
$A = (1; 5]$
$C = [2; 3]$

Russell’s Paradox

Suppose a set can be either *“normal”* or *“unusual”*.

- A set is considered *normal* if it does *not contain itself* as an element. That is, $A \notin A$.
- Otherwise, it is *unusual*. That is, $A \in A$.



Bertrand Russell

Note: being “normal” or “unusual” is a predicate $P(x)$ that can be applied to any set x.

Consider the set R of *all normal sets*: $R = \{A \mid A \notin A\}$.

The paradox arises when we ask: **Is $\boldsymbol{R}$ a normal set?**

- Suppose R is *normal*. By its definition, R must be an element of R, so $R \in R$. But elements of R are normal sets, and normal sets do not contain themselves. So $R \notin R$. Contradiction.
- Suppose R is *unusual*. This means R contains itself, so $R \in R$. But the definition of R only includes sets that do *not* contain themselves. So R cannot be a member of R, i.e. $R \notin R$. Contradiction.

A contradiction is reached in both cases. The only possible conclusion is that **the set $\boldsymbol{R}$ cannot exist**.

This paradox showed that *unrestricted comprehension* — the ability to form a set from any arbitrary property — is logically inconsistent.

From Naïve to Axiomatic Set Theory

Definition 12: *Naïve set theory* allows unrestricted set formation: for any property $P(x)$, we can form the set $\{x \mid P(x)\}$. Russell's paradox shows this leads to contradictions.

Definition 13: *Axiomatic set theory* restricts set formation through a system of axioms.

The most widely accepted system is *Zermelo-Fraenkel set theory with the Axiom of Choice* (ZFC).

ZFC Axioms

1. **Extensionality**: Sets with the same elements are equal.
2. **Empty Set**: There exists a set $\emptyset$ with no elements.
3. **Pairing**: For any a and b, there exists a set $\{a, b\}$.
4. **Union**: For any collection of sets, their union exists.
5. **Power Set**: For any set A, the power set $\mathcal{P}(A)$ exists.
6. **Infinity**: There exists an infinite set (containing $\mathbb{N}$).
7. **Separation**: From any set A and property P, we can form $\{x \in A \mid P(x)\}$.
8. **Replacement**: If F is a function-like relation, then for any set A, the image $F[A]$ exists.
9. **Foundation**: Every non-empty set has a minimal element (prevents self-membership).
10. **Choice**: Every collection of non-empty sets has a choice function.

Note: The **Separation** axiom prevents Russell's paradox by only allowing formation of subsets from existing sets, not arbitrary collections.

Relations

"In mathematics you don't understand things. You just get used to them."

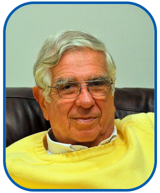
— John von Neumann

René Descartes

Évariste Galois

Ernst Schröder

Michael Rabin



Herbert Wilf

Relations as Sets

Definition 14: A *binary relation* R on sets A and B is a subset of the Cartesian product $A \times B$.

Notation: If $R \subseteq A \times B$, we write "$a\ R\ b$" to mean that element $a \in A$ is *related* to element $b \in B$.

Formally, $a\ R\ b$ iff $\langle a, b\rangle \in R$.

Note: R is used to denote both the relation itself ($a\ R\ b$) *and* the set of pairs ($R \subseteq A \times B$).

Note: the *order* of elements in the pair *matters*: $\langle a, b\rangle \in R$ denotes that a is related to b, not the other way around, unless there is *another* pair $\langle b, a\rangle$ in the relation.

Example: $R = \{\langle n, k\rangle \mid n, k \in \mathbb{N} \text{ and } n < k\}$

Definition 15:

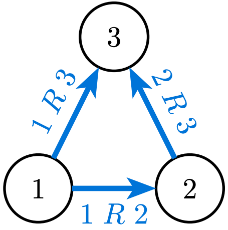
- A binary relation $R \subseteq A \times B$ on two different sets A and B is called *heterogeneous*.
- A binary relation $R \subseteq M^2$ on the same set M is called *homogeneous*.

Graph Representation

Definition 16: A homogeneous relation $R \subseteq M^2$ can be represented as a *directed graph* where:

- Vertices correspond to elements of M
- There is a directed edge from x to y if $x \mathrel{R} y$, i.e. $\langle x, y \rangle \in R$

Example: For $M = \{1, 2, 3\}$ and $R = \{\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 1, 3 \rangle\}$, the graph has vertices $\{1, 2, 3\}$ and directed edges $1 \to 2$, $2 \to 3$, and $1 \to 3$.



Matrix Representation

Definition 17: A binary relation $R \subseteq A \times B$ can be represented as a *matrix* $M_R = [\![R]\!]$ where:

- Rows correspond to elements of A
- Columns correspond to elements of B
- $M_R[i, j] = 1$ if $a_i \; R \; b_j$, and $M_R[i, j] = 0$ otherwise

Example: Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $R = \{\langle a, x\rangle, \langle b, x\rangle, \langle c, y\rangle\}$. The matrix representation is:

$$[\![R]\!] = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{where rows are } \{a, b, c\} \text{ and columns are } \{x, y\}$$

Special Relations

Definition 18: For any set M, we define these special relations:

- *Empty relation*: $\varnothing \subseteq M^2$ (no elements are related)
- *Identity relation*: $I_M = \{\langle x, x\rangle \mid x \in M\}$ (each element related only to itself)
- *Universal relation*: $U_M = M^2$ (every element related to every element)

Example: For $M = \{a, b, c\}$:

- Empty: $\varnothing$
- Identity: $\{\langle a, a\rangle, \langle b, b\rangle, (c, c)\}$
- Universal: $\{\langle a, a\rangle, \langle a, b\rangle, \langle a, c\rangle\langle b, a\rangle, \langle b, b\rangle, \langle b, c\rangle, \langle c, a\rangle, \langle c, b\rangle, \langle c, c\rangle\}$ (all 9 pairs)

Operations on Relations

Definition 19: For relations $R, S \subseteq A \times B$:

- *Union*: $R \cup S = \{\langle a, b\rangle \mid \langle a, b\rangle \in R \text{ or } \langle a, b\rangle \in S\}$
- *Intersection*: $R \cap S = \{\langle a, b\rangle \mid \langle a, b\rangle \in R \text{ and } \langle a, b\rangle \in S\}$
- *Complement*: $\overline{R} = (A \times B) \setminus R$

Definition 20: For a relation $R \subseteq A \times B$, the *converse* (or *inverse*) relation is:

$$R^{-1} = \{\langle b, a\rangle \mid \langle a, b\rangle \in R\} \subseteq B \times A$$

Example: If $R = \{\langle 1, x\rangle, \langle 2, y\rangle, \langle 2, z\rangle\}$, then $R^{-1} = \{\langle x, 1\rangle, \langle y, 2\rangle, \langle z, 2\rangle\}$.

Closures of Relations

Definition 21: Let $R \subseteq M^2$ be a relation. The *closures* of R are the smallest relations containing R with specific properties:

- *Reflexive closure*: $R^+ = R \cup I_M$
- *Symmetric closure*: $R^s = R \cup R^{-1}$
- *Transitive closure*: R^* is the smallest transitive relation containing R

Properties of Homogeneous Relations

Definition 22: A relation $R \subseteq M^2$ is *reflexive* if every element is related to itself:

$$\forall x \in M.\, (x\ R\ x)$$

Definition 23: A relation $R \subseteq M^2$ is *symmetric* if for every pair of elements, if one is related to the other, then the reverse is also true:

$$\forall x, y \in M.\, (x\ R\ y) \to (y\ R\ x)$$

Definition 24: A relation $R \subseteq M^2$ is *transitive* if for every three elements, if the first is related to the second, and the second is related to the third, then the first is also related to the third:

$$\forall x, y, z \in M.\, (x\ R\ y \land y\ R\ z) \to (x\ R\ z)$$

More Properties

Definition 25: A relation $R \subseteq M^2$ is *irreflexive* if no element is related to itself:

$$\forall x \in M.\, (x \not{R}\, x)$$

Definition 26: A relation $R \subseteq M^2$ is *antisymmetric* if for every pair of elements, if both are related to each other, then they must be equal:

$$\forall x, y \in M.\, (x \mathrel{R} y \land y \mathrel{R} x) \to (x = y)$$

Definition 27: A relation $R \subseteq M^2$ is *asymmetric* if for every pair of elements, if one is related to the other, then the reverse is not true:

$$\forall x, y \in M.\, (x \mathrel{R} y) \to (y \not{R}\, x)$$

Note: *irreflexive* + *antisymmetric* = *asymmetric*.

Additional Properties

Definition 28: A relation $R \subseteq M^2$ is:

- *Coreflexive* if $R \subseteq I_M$ (only related to themselves, if at all):

$$\forall x, y \in M.\, (x \mathrel{R} y) \to (x = y)$$

- *Left Euclidean* if whenever an element is related to two others, those two are related:

$$\forall x, y, z \in M.\, (x \mathrel{R} y \land x \mathrel{R} z) \to (y \mathrel{R} z)$$

- *Right Euclidean* if whenever two elements are both related to a third, they are related to each other:

$$\forall x, y, z \in M.\, (y \mathrel{R} x \land z \mathrel{R} x) \to (y \mathrel{R} z)$$

Example:

- Identity relation I_M is coreflexive. Any subset of I_M is also coreflexive.
- Equality relation “$=$” is left and right Euclidean.
- “Being in the same equivalence class” is Euclidean in both directions.

Equivalence Relations

Definition 29: A relation $R \subseteq M^2$ is an *equivalence relation* if it is reflexive, symmetric and transitive.

Definition 30: Let $R \subseteq M^2$ be an equivalence relation on a set M. The *equivalence class* of an element $x \in M$ under R is the set of all elements related to x:

$$[x]_R = \{y \in M \mid x \mathrel{R} y\}$$

Definition 31: The *quotient set* of M by the equivalence relation R is the set of all equivalence classes:

$$M/_R = \{[x]_R \mid x \in M\}$$

Theorem 2: If $R \subseteq M^2$ is an equivalence relation, then $x \mathrel{R} y$ iff $[x]_R = [y]_R$ for all $x, y \in M$.

Partitions

Definition 32: A *partition* $\mathcal{P}$ of a set M is a family of non-empty, pairwise-disjoint subsets whose union is M:

- (Non-empty) $\forall B \in \mathcal{P}.\,(B \neq \varnothing)$
- (Disjoint) $\forall B_1, B_2 \in \mathcal{P}.\,(B_1 \neq B_2) \to (B_1 \cap B_2 = \varnothing)$
- (Cover) $\bigcup\limits_{B \in \mathcal{P}} B = M$

Elements of $\mathcal{P}$ are *blocks* (or *cells*).

Example: For $M = \{0, 1, 2, 3, 4, 5\}$: $\{\{0, 2, 4\}, \{1, 3, 5\}\}$ (even / odd) and $\{\{0, 5\}, \{1, 2, 3\}, \{4\}\}$ (arbitrary) are partitions.

Partitions and Equivalence Relations

Theorem 3 (Equivalences ⇔ Partitions): Each equivalence relation R on M yields the partition $\mathcal{P}_R = \{[x]_R \mid x \in M\}$. Each partition $\mathcal{P}$ yields an equivalence $R_{\mathcal{P}}$ given by $x R_{\mathcal{P}} y$ iff x, y lie in the same block. These constructions invert one another.

Proof *(Sketch)*: Classes of an equivalence are non-empty, disjoint, and cover M. Conversely "same block" relation is reflexive, symmetric, transitive. Composing the two constructions returns exactly the starting equivalence relation or partition (they are mutually inverse up to equality of sets of ordered pairs). □

Orders

Definition 33: A relation $R \subseteq M^2$ is called a *preorder* if it is reflexive and transitive.

Definition 34: A *partial order* is a relation $R \subseteq M^2$ that is reflexive, antisymmetric, and transitive.

Definition 35: A relation $R \subseteq M^2$ is *connected* if for every pair of distinct elements, either one is related to the other or vice versa:

$$\forall x, y \in M.\, (x \neq y) \to (x \mathrel{R} y \lor y \mathrel{R} x)$$

Definition 36: A partial order which is also connected is called a *total order* (or *linear order*).

Chains and Antichains

Definition 37: In a partially ordered set $(M, \preccurlyeq)$:

- A *chain* is a subset $C \subseteq M$ where every two elements are comparable. Formally:

$$\forall x, y \in C.\, (x \preccurlyeq y \text{ or } y \preccurlyeq x)$$

- An *antichain* is a subset $A \subseteq M$ where no two distinct elements are comparable. Formally:

$$\forall x, y \in A.\, (x \neq y) \to (x \not\preccurlyeq y \text{ and } y \not\preccurlyeq x)$$

Example: Consider the divisibility relation $|$ on $\{1, 2, 3, 4, 6, 12\}$:

- Chain: $\{1, 2, 4, 12\}$ (since $1 \mid 2 \mid 4 \mid 12$)
- Chain: $\{1, 3, 6, 12\}$ (since $1 \mid 3 \mid 6 \mid 12$)
- Antichain: $\{2, 3\}$ (since $2 \nmid 3$ and $3 \nmid 2$)
- Antichain: $\{4, 6\}$ (since $4 \nmid 6$ and $6 \nmid 4$)

Dilworth's Theorem

Theorem 4 (Dilworth): In any finite partially ordered set, the maximum size of an antichain equals the minimum number of chains needed to cover the entire set.

Example: In the Boolean lattice $\mathcal{P}(\{a, b\})$ with inclusion:

- Maximum antichain: $\{\{a\}, \{b\}\}$ of size 2
- Minimum chain decomposition: $\{\emptyset, \{a\}\} \cup \{\{b\}, \{a, b\}\}$ with 2 chains

Examples of Orders

Example: Consider the *no longer than* relation $\preccurlyeq$ on $\mathbb{B}^*$: $x \preccurlyeq y$ iff $\operatorname{len}(x) \leq \operatorname{len}(y)$. This is a preorder (reflexive and transitive), and even connected, but not a partial order, since it is not antisymmetric: for example, $01 \preccurlyeq 10$ and $10 \preccurlyeq 01$, but $01 \neq 10$.

Example: The subset relation $\subseteq$ on $\mathcal{P}(A)$ is a partial order (reflexive, antisymmetric, transitive); typically not total, since not all subsets are comparable (e.g., $A = \{1\}$ and $B = \{2, 3\}$).

Example: Divisibility $|$ on $D = \{1, 2, 3, 6\}$: $1|2|6$, $1|3|6$; 2 and 3 incomparable. Partial, not total.

Example: Lexicographic order on A^n (induced by a total order on A) is a total order.

Composition of Relations

Definition 38: The *composition* of two relations $R \subseteq A \times B$ and $S \subseteq B \times C$ is defined as:

$$R \mathbin{;} S = S \circ R = \{\langle a, c \rangle \mid \exists b \in B.\, (a \mathrel{R} b) \land (b \mathrel{S} c)\}$$

Powers of Relations

Definition 39: For a homogeneous relation $R \subseteq M^2$, we define *powers* of R:

- $R^0 = I_M$ (identity relation)
- $R^1 = R$
- $R^{n+1} = R^n \circ R$ for $n \geq 1$

Example: Let $M = \{1, 2, 3, 4\}$ and $R = \{\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 4 \rangle\}$ (successor relation).

- $R^1 = \{\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 4 \rangle\}$
- $R^2 = \{\langle 1, 3 \rangle, \langle 2, 4 \rangle\}$ (two steps)
- $R^3 = \{\langle 1, 4 \rangle\}$ (three steps)
- $R^4 = \varnothing$ (no four-step paths)

Theorem 5: For any relation R on a finite set with n elements:

- $R^+ = R^1 \cup R^2 \cup \ldots \cup R^n$ is a *transitive closure*.
- $R^* = R^0 \cup R^+ = I \cup R^+$ is a *reflexive-transitive closure*.

Associativity of Composition

Theorem 6: Composition of relations is associative: $(R \mathbin{;} S) \mathbin{;} T = R \mathbin{;} (S \mathbin{;} T)$.

Proof: Let $R \subseteq A \times B$, $S \subseteq B \times C$, and $T \subseteq C \times D$ be three relations.

($\subseteq$): Let $\langle a, d \rangle \in (R \mathbin{;} S) \mathbin{;} T$.

- By definition of composition: $\exists c \in C.\, (\langle a, c \rangle \in R \mathbin{;} S) \land (\langle c, d \rangle \in T)$.
- Since $\langle a, c \rangle \in (R \mathbin{;} S)$, we have: $\exists b \in B.\, (\langle a, b \rangle \in R) \land (\langle b, c \rangle \in S)$.
- From $\langle b, c \rangle \in S$ and $\langle c, d \rangle \in T$, we have: $\langle b, d \rangle \in S \mathbin{;} T$.
- From $\langle a, b \rangle \in R$ and $\langle b, d \rangle \in S \mathbin{;} T$, we have: $\langle a, d \rangle \in R \mathbin{;} (S \mathbin{;} T)$.

($\supseteq$): Let $\langle a, d \rangle \in R \mathbin{;} (S \mathbin{;} T)$.

- By definition of composition: $\exists b \in B.\, (\langle a, b \rangle \in R) \land (\langle b, d \rangle \in S \mathbin{;} T)$.
- Since $\langle b, d \rangle \in S \mathbin{;} T$, we have: $\exists c \in C.\, (\langle b, c \rangle \in S) \land (\langle c, d \rangle \in T)$.
- From $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in S$, we have: $\langle a, c \rangle \in R \mathbin{;} S$.
- From $\langle a, c \rangle \in R \mathbin{;} S$ and $\langle c, d \rangle \in T$, we have: $\langle a, d \rangle \in (R \mathbin{;} S) \mathbin{;} T$.

Therefore, $(R \mathbin{;} S) \mathbin{;} T = R \mathbin{;} (S \mathbin{;} T)$. □

Functions

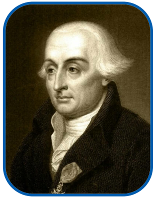
"A function is a machine which converts a certain class of inputs into a certain class of outputs."

— Norbert Wiener

Leonhard Euler

Augustin-Louis Cauchy

Karl Weierstrass



Joseph-Louis Lagrange

George Pólya

Norbert Wiener

Definition of a Function

Definition 40: A *function* f from a set A to a set B, denoted $f : A \to B$, is a special kind of relation $f \subseteq A \times B$ where every element of A is paired with *exactly one* element of B.

This means two conditions must hold:

1. *Functional (left-total)*: For every $a \in A$, there is *at least one* pair $\langle a, b \rangle$ in f.

$$\forall a \in A, \exists b \in B : \langle a, b \rangle \in f$$

2. *Serial (right-unique)*: For every $a \in A$, there is *at most one* pair $\langle a, b \rangle$ in f.

$$(\langle a, b_1 \rangle \in f \land \langle a, b_2 \rangle \in f) \Longrightarrow b_1 = b_2$$

Definition 41: A relation that satisfies the functional (left-total) property is called a *partial function*.

A relation that satisfies *both* properties is called a *total function*, or simply a *function*.

Domain, Codomain, Range

Definition 42: For a function $f : A \to B$:

- The set A is called the *domain* of f, denoted $\operatorname{Dom}(f)$.
- The set B is called the *codomain* of f, denoted $\operatorname{Cod}(f)$.
- The *range* (or *image*) of f is the set of all values that f actually takes:

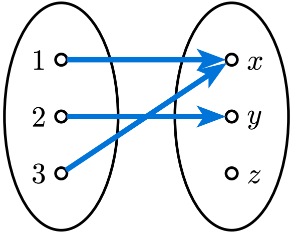
$$\operatorname{Range}(f) = \{b \in B \mid \exists a \in A, f(a) = b\} = \{f(a) \mid a \in A\}$$

Note: $\operatorname{Range}(f) \subseteq \operatorname{Cod}(f)$

Example: Let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$. Let $f = \{\langle 1, x\rangle, \langle 2, y\rangle, \langle 3, x\rangle\}$.

- f is a function from A to B.
- $\operatorname{Dom}(f) = A$
- $\operatorname{Cod}(f) = B$
- $\operatorname{Range}(f) = \{x, y\} \subseteq B$

We have $f(1) = x$, $f(2) = y$, $f(3) = x$.



Domain, Codomain, Range [2]

Example: Consider $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(n) = n^2$.

- $\mathrm{Dom}(g) = \mathbb{Z}$.
- $\mathrm{Cod}(g) = \mathbb{Z}$.
- $\mathrm{Range}(g) = \{0, 1, 4, 9, ...\}$ (the set of non-negative perfect squares).

Injective Functions

Definition 43: A function $f : A \to B$ is *injective* (or *one-to-one*[3]) if distinct elements in the domain map to distinct elements in the codomain.

$$\forall a_1, a_2 \in A, (f(a_1) = f(a_2)) \Longrightarrow (a_1 = a_2)$$

Example: $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$ is injective. If $f(n_1) = f(n_2)$, then $2n_1 = 2n_2$, so $n_1 = n_2$.

Example: $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(n) = n^2$ is *not* injective, because $g(-1) = 1$ and $g(1) = 1$, but $-1 \neq 1$.

[3] Do not confuse it with *one-to-one correspondence*, which is a bijection, not just injection!

Surjective Functions

Definition 44: A function $f : A \to B$ is *surjective* (or *onto*) if every element in the codomain is the image of at least one element in the domain.

$$\forall b \in B.\, \exists a \in A.\, f(a) = b$$

For surjective functions, $\mathrm{Range}(f) = \mathrm{Cod}(f)$.

Example: $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is surjective. For any $y \in \mathbb{R}$, $x = \sqrt[3]{y}$ is such that $f(x) = y$.

Example: $g : \mathbb{N} \to \mathbb{N}$ defined by $g(n) = 2n$ is *not* surjective, because odd numbers in $\mathbb{N}$ (the codomain) are not in the range of g. For example, there is no $n \in \mathbb{N}$ such that $2n = 3$.

Bijective Functions

Definition 45: A function $f : A \to B$ is *bijective* if it is both injective and surjective. A bijective function establishes a *one-to-one correspondence* between the elements of A and B.

Example: $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$ is bijective.

- Injective: If $2x_1 + 1 = 2x_2 + 1$, then $2x_1 = 2x_2$, so $x_1 = x_2$.
- Surjective: For any $y \in \mathbb{R}$, let $x = \frac{y-1}{2}$. Then $f(x) = 2\left(\frac{y-1}{2}\right) + 1 = y - 1 + 1 = y$.

Example: The identity function $\mathrm{id}_A : A \to A$ defined by $\mathrm{id}_{A(x)} = x$ for all $x \in A$ is bijective.

Function Composition

Definition 46: Let $f : A \to B$ and $g : B \to C$ be two functions. The *composition* of g and f, denoted $g \circ f$ (read as “g composed with f” or “g after f”), is a function from A to C defined by:

$$(g \circ f)(a) = g(f(a))$$

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$ be $g(x) = x + 1$.

- $(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 1$.
- $(f \circ g)(x) = f(g(x)) = f(x + 1) = (x + 1)^2 = x^2 + 2x + 1$.

Properties of Function Composition

- *Associativity:* If $f : A \to B$, $g : B \to C$, and $h : C \to D$, then $(h \circ g) \circ f = h \circ (g \circ f)$.

- The *identity* function acts as a *neutral* element for composition:
    - $\mathrm{id}_B \circ f = f$ for any function $f : A \to B$.
    - $f \circ \mathrm{id}_A = f$ for any function $f : A \to B$.

- Composition *preserves* the properties of functions:
    - If f and g are injective, so is $g \circ f$.
    - If f and g are surjective, so is $g \circ f$.
    - If f and g are bijective, so is $g \circ f$.

- Note that in general, $g \circ f \neq f \circ g$, i.e., function composition is *not commutative*.

Inverse Functions

Definition 47: If $f : A \to B$ is a bijective function, then its *inverse function*, denoted $f^{-1} : B \to A$, is defined as:

$$f^{-1}(b) = a \quad \text{iff} \quad f(a) = b$$

Note: A function has an inverse *if and only if* it is bijective.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = 2x + 1$. We found it's bijective. To find $f^{-1}(y)$, let $y = 2x + 1$. Solving for x, we get $x = \frac{y-1}{2}$. So, $f^{-1}(y) = \frac{y-1}{2}$.

Theorem 7: If $f : A \to B$ is a bijective function with inverse $f^{-1} : B \to A$:

- f^{-1} is also bijective.
- $(f^{-1} \circ f)(a) = a$ for all $a \in A$ (i.e., $f^{-1} \circ f = \mathrm{id}_A$).
- $(f \circ f^{-1})(b) = b$ for all $b \in B$ (i.e., $f \circ f^{-1} = \mathrm{id}_B$).
- If $f : A \to B$ and $g : B \to C$ are both bijective, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Image and Preimage of Sets

Definition 48: Let $f : A \to B$ be a function and let $S \subseteq A$. The *image of S under f* is the set:

$$f(S) = \{f(s) \mid s \in S\}$$

Note that $f(S) \subseteq B$. The range of f is $f(A)$.

Definition 49: Let $f : A \to B$ be a function and let $T \subseteq B$. The *preimage of T under f* (or *inverse image of T*) is the set of all elements in the domain that map into T:

$$f^{-1}(T) = \{a \in A \mid f(a) \in T\}$$

Note: The notation $f^{-1}(T)$ is used even if the inverse function f^{-1} does not exist (i.e., if f is not bijective). It always refers to the set of domain elements that map into T.

Image and Preimage of Sets [2]

Example: Let $f : \mathbb{Z} \to \mathbb{Z}$ be $f(x) = x^2$.

- Let $S = \{-2, -1, 0, 1, 2\}$. Then $f(S) = \{f(-2), f(-1), f(0), f(1), f(2)\} = \{4, 1, 0, 1, 4\} = \{0, 1, 4\}$.
- Let $T_1 = \{1, 9\}$. The preimage is $f^{-1}(T_1) = \{x \in \mathbb{Z} \mid x^2 \in \{1, 9\}\} = \{-3, -1, 1, 3\}$.
- Let $T_2 = \{2, 3\}$. The preimage is $f^{-1}(T_2) = \{x \in \mathbb{Z} \mid x^2 \in \{2, 3\}\} = \varnothing$.

Cardinality & Infinity

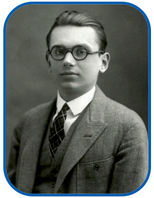
"God made the integers, all else is the work of man."

— Leopold Kronecker

Giuseppe Peano

Leopold Kronecker

David Hilbert



Kurt Gödel

John von Neumann

Size of Sets

Definition 50: The *size* of a *finite* set X, denoted $|X|$, is the number of elements it contains.

Examples:

- Let $A = \{$🪐 , 🦕 , 🎻 $\}$, then $|A| = 3$, since A contains *exactly 3* elements.
- Let $B = \{$🥝 , 🥝 , 🥝 $\}$, then $|B| = 1$, since B contains *only one unique* element (the kiwi).
- $|\mathcal{P}(\{1, 2,$ 🐈 $\})| = 2^3 = 8$, since the power set consists of *all 8 possible subsets* of $\{1, 2,$ 🐈 $\}$.
- $|\varnothing| = 0$, since the *empty* set contains *no elements*.
- $|\mathbb{N}| = \infty$, since there are *infinitely many* natural numbers.
- $|\mathbb{R}| = \infty$, since there are *infinitely many* real numbers.

Cardinality of Sets

Definition 51: Two sets A and B have the same *cardinality* and called *equinumerous*, denoted $|A| = |B|$ or $A \approx B$, iff there is a *bijection* (one-to-one correspondence) from A to B.

Theorem 8: Equinumerosity is an equivalence relation.

Proof: Let A, B, C be sets.

- *Reflexivity:* The identity map $\mathrm{id}_A : A \to A$, where $\mathrm{id}_A(x) = x$, is a bijection, so $A \approx A$.
- *Symmetry:* Suppose $A \approx B$, then there is a bijection $f : A \to B$. Since it is a bijection, its inverse f^{-1} exists and is also a bijection. Hence, $f^{-1} : B \to A$ is a bijection, so $B \approx A$.
- *Transitivity:* Suppose that $A \approx B$ and $B \approx C$, i.e., there are bijections $f : A \to B$ and $g : B \to C$. Then the composition $g \circ f : A \to C$ is also a bijection. So $A \approx C$.

□

Countable Sets

Definition 52: A set called *countable* if it is either finite or has the same cardinality as the set of natural numbers $\mathbb{N}$. Alternatively, a set is countable if there is a *bijection* from $\mathbb{N}$ to that set.

When an infinite set is *countable*, its cardinality is denoted $\aleph_0$ (*"aleph-null"*).

Example: $|\mathbb{N}_{\text{odd}} = \{1, 3, 5, ...\}| = \aleph_0$, the set of *odd* natural numbers is countable, since there is a bijection $f : \mathbb{N} \to \mathbb{N}_{\text{odd}}$ defined by $f(n) = 2n + 1$.

Example: $|\{x \in \mathbb{N} \mid x \text{ is prime}\}| = \aleph_0$, the set of *prime* numbers is countable.

Example: $|\mathbb{Z}| = \aleph_0$, the set of *integers* $(-\infty, ..., -2, -1, 0, 1, 2, ..., \infty)$ is countable, since there is a bijection $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(n)$:

$$f(n) = (-1)^n \left\lceil \frac{n}{2} \right\rceil = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases} \qquad \begin{bmatrix} f(0) & f(1) & f(2) & f(3) & f(4) & f(5) & f(6) & \dots \\ \lceil \frac{0}{2} \rceil & -\lceil \frac{1}{2} \rceil & \lceil \frac{2}{2} \rceil & -\lceil \frac{3}{2} \rceil & \lceil \frac{4}{2} \rceil & -\lceil \frac{5}{2} \rceil & \lceil \frac{6}{2} \rceil & \dots \\ 0 & -1 & 1 & -2 & 2 & -3 & 3 & \dots \end{bmatrix}$$

Countability Constructions

Definition 53: A set X is *enumerable* if there is a surjection $e : \mathbb{N} \to X$ (equivalently a bijection with either $\mathbb{N}$ or an initial segment of $\mathbb{N}$ if X finite).

Theorem 9 (Zig-Zag Enumeration): $\mathbb{N}^2$ is countable.

Proof: List pairs by diagonals of constant sum: $\langle 0, 0\rangle$; $\langle 0, 1\rangle, \langle 1, 0\rangle$; $\langle 0, 2\rangle, \langle 1, 1\rangle, \langle 2, 0\rangle$; … giving a bijection with $\mathbb{N}$. □

Theorem 10: $\mathbb{Q}$ is countable.

Proof: Enumerate positive reduced fractions p/q ordered by $p + q$ and increasing p; skip non-reduced. Interleave 0 and negatives. This yields *enumeration*, hence $\mathbb{Q} \approx \mathbb{N}$. □

Pairing Functions

Definition 54: A function $f : A \times B \to \mathbb{N}$ is an arithmetical *pairing function* if it is injective.

We say that f *encodes* $A \times B$, and that $f(a, b)$ is the *code* of the pair $\langle a, b \rangle$.

Example: The *Cantor pairing function* $g : \mathbb{N}^2 \to \mathbb{N}$ is defined as:

$$g(n, k) = \frac{(n + k + 1)(n + k)}{2} + n$$

Georg Cantor

Uncountable Sets

Definition 55: A set is *uncountable* if it is not countable.

In order to prove that a set A is *uncountable*, we need to show that *no bijection* $\mathbb{N} \to A$ *can exist*.

The general strategy for showing that is to use *Cantor's diagonal argument*. Given a list of elements of A, say $x_1, x_2, \ldots$ (enumerated by natural numbers), we construct a *new* element of A that *differs* from each x_i, thus showing that the list cannot be complete, and hence no bijection can exist.

Theorem 11: B^ω is uncountable.

Proof: Recall that $\mathbb{B}^\omega$ is the set of all *infinite sequences* of elements from $\mathbb{B} = \{0, 1\}$.
For example, $\mathbb{B}^\omega$ contains sequences like 0000…, 010101…, 1110…, etc.

Suppose for contradiction that $\mathbb{B}^\omega$ is countable. Then we can *enumerate* its elements as $x_1, x_2, \ldots$, where each x_i is an infinite sequence of bits, so we can represent it as $x_i = (b_{i1}, b_{i2}, b_{i3}, \ldots)$, where $b_{ij} \in \mathbb{B}$ is the j-th bit of the i-th sequence.

Uncountable Sets [2]

Now we construct a new sequence $\Delta = \left(\overline{b}_{11}, \overline{b}_{22}, \overline{b}_{33}, \ldots\right)$, where $\overline{b}_{ii} = 1 - b_{ii}$, i.e., we flip the i-th bit of the i-th sequence. This sequence *differs* from each x_i at least in the i-th position, so it cannot be equal to any x_i, so it is *not in* the enumeration $x_1, x_2, \ldots$.

	1	2	3	…
x_1	$\boldsymbol{b_{11}}$	b_{12}	b_{13}	$\ldots$
x_2	b_{21}	$\boldsymbol{b_{22}}$	b_{23}	$\ldots$
x_3	b_{31}	b_{32}	$\boldsymbol{b_{33}}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
Δ	$\overline{b}_{11}$	$\overline{b}_{22}$	$\overline{b}_{33}$	$\ldots$

Since Δ is constructed from the bits, it is also an *element* of $\mathbb{B}^{\omega}$. Thus, we have found an element of $\mathbb{B}^{\omega}$ that is not in the enumeration $x_1, x_2, \ldots$, contradicting the assumption that $\mathbb{B}^{\omega}$ is countable. □

Sets of Different Sizes

Definition 56: The cardinality of a set A is *less than or the same* as the cardinality of a set B, denoted $|A| \leq |B|$ or $A \preceq B$, if there is an *injection* (one-to-one function) from A to B.

Definition 57: Set A is *smaller* than B, denoted $|A| < |B|$ or $A \prec B$, iff there is an *injection*, but *no bijection* from A to B, i.e., $A \preceq B$ and $A \not\approx B$.

Note: Using this notation, we can say that a set X is *countable* iff $X \preceq \mathbb{N}$, and *uncountable* iff $\mathbb{N} \prec X$.

Example: $\{1, 2\} \prec \{a, b, c\}$, since there is an injection $f : \{1, 2\} \to \{a, b, c\}$ defined by $f(1) = a$ and $f(2) = b$, but no bijection exists.

Example: $\mathbb{N} \preceq \mathbb{Z}$, since there is bijection (and thus an injection) $f : \mathbb{N} \to \mathbb{Z}$.

Example: $\mathbb{Z} \preceq \mathbb{N}$, since there is bijection (and thus an injection) $f : \mathbb{Z} \to \mathbb{N}$.

Example: $\mathbb{N} \prec \mathcal{P}(\mathbb{N})$, since there is an injection $f(x) = \{x\}$, but no bijection exists.

Cantor's Theorem

Theorem 12 (Cantor): $A \prec \mathcal{P}(A)$, for any set A.

Proof: The map $f(x) = \{x\}$ is an injection $f : A \to \mathcal{P}(A)$, since if $x \neq y$, then also $\{x\} \neq \{y\}$ by extensionality, and so $f(x) \neq f(y)$. So we have that $A \preceq \mathcal{P}(A)$.

It remains to show that $A \not\approx B$. For reductio, suppose $A \approx B$, i.e., there is some bijection $g : A \to B$. Now consider $D = \{x \in A \mid x \notin g(x)\}$. Note that $D \subseteq A$, so $D \in \mathcal{P}(A)$. Since g is a bijection, there exists some $y \in A$ such that $g(y) = D$. But now we have

$$y \in g(y) \text{ iff } y \in D \text{ iff } y \notin g(y)$$

This is a contradiction, since y cannot be both *in* and *not in* $g(y)$. Thus, $A \not\approx \mathcal{P}(A)$. □

Schröder–Bernstein Theorem

Theorem 13 (Schröder–Bernstein): If $A \preceq B$ and $B \preceq A$, then $A \approx B$.

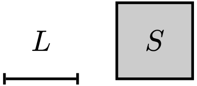
In other words, if there are injections in both directions between two sets, then there is a bijection.

Proof: Obvious, but difficult. 🤷 □

Another Cantor's Theorem

Let L be the unit line, i.e., the set of points $[0, 1]$. Let S be the unit square, i.e., the set of points $L \times L$.

Theorem 14: $L \approx S$.



Proof[4]: Consider the function $f : L \to S$ defined by $f(x) = (x, x)$. This is an injection, since if $f(a) = f(b)$, then $(a, a) = (b, b)$, so $a = b$. Thus, $L \preceq S$.

Now consider the function $g : S \to L$ that maps (x, y) to the real number obtained by *interleaving* the decimal expansions of x and y.

$$\left.\begin{aligned} x &= 0.x_1 x_2 x_3 \ldots \\ y &= 0.y_1 y_2 y_3 \ldots \end{aligned}\right\} \quad g(x, y) = 0.x_1 y_1 x_2 y_2 x_3 y_3 \ldots$$

This is an injection, since if $g(a, b) = g(c, d)$, then $a_n = c_n$ and $b_n = d_n$ for all $n \in \mathbb{N}$, so $(a, b) = (c, d)$. Thus, $S \preceq L$.

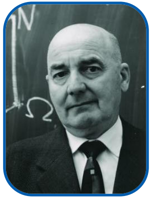
By Schröder–Bernstein (Theorem 13), we have that $L \approx S$. □

[4] See https://math.stackexchange.com/a/183383 for more detailed analysis.

Order Theory

"Order is heaven's first law."

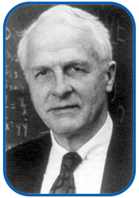
— Alexander Pope



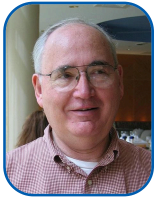
Helmut Hasse

Alfred Tarski

Emmy Noether



Garrett Birkhoff



Dana Scott

Felix Hausdorff

Partially Ordered Sets

Definition 58: A *partially ordered set* (or *poset*) $\langle S, \leq \rangle$ is a set S equipped with a partial order $\leq$.

Definition 59: A *chain* in a poset $\langle S, \leq \rangle$ is a subset $C \subseteq S$ such that any two elements $x, y \in C$ are *comparable*, i.e., either $x \leq y$ or $y \leq x$.

Definition 60: An element $x \in S$ is called a *minimal element* of a poset $\langle S, \leq \rangle$ if there is no "greater" element $y \in S$ such that $y < x$ (i.e., $y \leq x$ and $y \neq x$).

Definition 61: A *maximal element* m satisfies: there is no $y \in S$ with $m < y$.

Note: There may be multiple maximal (or minimal) elements.

Partially Ordered Sets [2]

Definition 62: The *greatest element* of a poset $\langle S, \leq \rangle$ is an element $g \in S$ that is greater than or equal to every other element in S, i.e., for all $x \in S$, $x \leq g$.

Definition 63: A *least element* (bottom) b satisfies $b \leq x$ for all $x \in S$.

Note: Greatest (top) and least (bottom) elements are *unique* when they exist.

Examples:

- $\langle \mathcal{P}(A), \subseteq \rangle$: least $\varnothing$, greatest A.
- $\langle \mathbb{N}^+, | \rangle$: least 1, no greatest element.
- $\langle \mathbb{Z}, \leq \rangle$: no least or greatest element.
- $\langle \{1, ..., 6\}, | \rangle$: least 1, no greatest element, maximal elements are 4, 5, 6.

Upper and Lower Bounds

Definition 64: In a poset $\langle S, \leq \rangle$, an element $u \in S$ is called an *upper bound* of a subset $C \subseteq S$ if it is greater than or equal to every element in C, i.e., for all $x \in C$, $x \leq u$.

Definition 65: In a poset $\langle S, \leq \rangle$, an element $l \in S$ is called a *lower bound* of a subset $C \subseteq S$ if it is less than or equal to every element in C, i.e., for all $x \in C$, $l \leq x$.

Examples:

- In $\langle \mathbb{R}, \leq \rangle$ for interval $C = (0, 1)$: every $x \leq 0$ is a lower bound; every $x \geq 1$ an upper bound.
- In $\langle \mathcal{P}(A), \subseteq \rangle$ for $C = \{\{1, 2\}, \{1, 3\}\}$: lower bounds include $\{1\}$, $\varnothing$; upper bounds include $\{1, 2, 3\}$.
- In $\langle \mathbb{Z}, | \rangle$ for $C = \{4, 6\}$: upper bounds are multiples of 12; least upper bound 12; lower bounds are divisors of 2; greatest lower bound 2.

Suprema and Infima

Definition 66: In a poset $\langle S, \leq \rangle$, the *supremum* (or *join*) of a subset $C \subseteq S$, denoted $\sup(C)$ or $\bigvee C$, is the *least upper bound* of C, i.e., an upper bound $u \in S$ s.t. for any other upper bound $v \in S$, $u \leq v$.

Note: If it exists, the least upper bound is *unique*.

Definition 67: In a poset $\langle S, \leq \rangle$, the *infimum* (or *meet*) of a subset $C \subseteq S$, denoted $\inf(C)$ or $\bigwedge C$, is the *greatest lower bound* of C, i.e., a lower bound $l \in S$ s.t. for any other lower bound $m \in S$, $m \leq l$.

Note: If it exists, the greatest lower bound is *unique*.

Examples:

- $\langle \mathbb{R}, \leq \rangle$: $\sup(\{0, 1\}) = 1$, $\inf(\{0, 1\}) = 0$, i.e., $\sup(C) = \max(C)$, $\inf(C) = \min(C)$.
- $\langle \mathcal{P}(A), \subseteq \rangle$: $\sup = \cup$, $\inf = \cap$.
- Divisibility on $\mathbb{N}_{>0}$: $\sup\{a, b\} = \operatorname{lcm}(a, b)$ (if any common multiple), $\inf\{a, b\} = \gcd(a, b)$.

Lattices

Definition 68: A poset $\langle S, \leq \rangle$ where every non-empty finite subset $C \subseteq S$ has a join (supremum) is called an *upper semilattice* (or *join-semilattice*) and denoted $\langle S, \vee \rangle$.

Definition 69: A poset $\langle S, \leq \rangle$ where every non-empty finite subset $C \subseteq S$ has a meet (infimum) is called a *lower semilattice* (or *meet-semilattice*) and denoted $\langle S, \wedge \rangle$.

Definition 70: A poset $\langle S, \leq \rangle$ that is both an upper semilattice and a lower semilattice, i.e., every non-empty finite subset has both a join and a meet, is called a *lattice*, denoted $(S, \vee, \wedge)$.

Why Lattices?

Why study lattices? Whenever you have:

- Elements that can be *compared* (ordered)
- Ways to *combine* elements (join, meet)
- Consistent behavior under combination

…you likely have a lattice! This structure appears in programming languages, databases, security systems, logic circuits, and many other areas of computer science and mathematics.

Properties of Lattices

Definition 71: A lattice is *bounded* if it has a greatest element $\top$ and a least element $\bot$.

Definition 72: A lattice is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (and dually).

Definition 73: A lattice is *modular* if $x \leq z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Note: Distributive $\Rightarrow$ modular.

Example (Powerset Lattice): $\langle \mathcal{P}(A), \subseteq \rangle$ is a bounded distributive lattice with $\vee = \cup$, $\wedge = \cap$, $\top = A$, $\bot = \emptyset$.

Why this matters: This is the foundation of set-based reasoning in:

- Database theory (relational algebra)
- Formal specification languages (Z, B-method)
- Model checking and verification

Examples of Lattices

Example (Divisibility Lattice): For positive integers, $a \leq b$ iff a divides b.

- Join: Least Common Multiple (LCM)
- Meet: Greatest Common Divisor (GCD)
- Used in: Number theory, cryptography (RSA), computer algebra systems

Example (Partition Lattice): All partitions of a set S, ordered by refinement.

- $\pi_1 \leq \pi_2$ if π_1 is a refinement of π_2 (smaller blocks)
- Join: Coarsest common refinement
- Meet: Finest common coarsening
- Applications: Clustering, database normalization

Lattices aren't just abstract algebra — they appear everywhere in computer science and mathematics.

The *join* and *meet* operations capture fundamental patterns of *combination* and *interaction*.

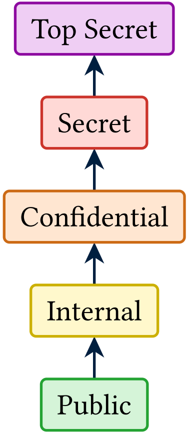
Why Lattices Matter [1]: Information Security Levels

Example: In computer security, information has *classification levels* forming a lattice:

- Elements: $\{\text{Public}, \text{Internal}, \text{Confidential}, \text{Secret}, \text{Top Secret}\}$
- Order: $\text{Public} \leq \text{Internal} \leq \text{Confidential} \leq \text{Secret} \leq \text{Top Secret}$
- Join ($\vee$): Higher classification needed to combine information
- Meet ($\wedge$): Lower classification that both pieces can be declassified to

For instance:

- $\text{Internal} \vee \text{Confidential} = \text{Confidential}$ (combination needs higher level)
- $\text{Secret} \wedge \text{Confidential} = \text{Confidential}$ (both can be declassified to this level)



Why Lattices Matter [2]: Program Analysis and Type Systems

Example: In programming language theory, *types* form lattices:

Subtype Lattice:

- Order: `int` $\sqsubseteq$ `number` $\sqsubseteq$ `any`, `string` $\sqsubseteq$ `any`
- Join: Most general common supertype (for union types)
- Meet: Most specific common subtype (for intersection types)

Control Flow Analysis:

- Elements: Sets of possible program states
- Order: Subset inclusion ($\subseteq$)
- Join: Union of possible states (at merge points)
- Meet: Intersection of guaranteed properties

Why Lattices Matter [3]: Database Query Optimization

Example: *Query execution plans* form a lattice:

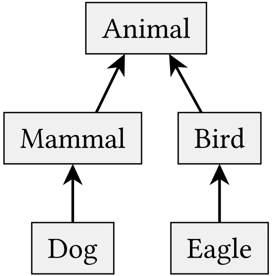
- Elements: Different ways to execute a query
- Order: “Plan A $\leq$ Plan B” if A is more efficient than B
- Join: Combine optimization strategies
- Meet: Find common optimizations

This structure helps database optimizers systematically explore the space of possible query plans.

Why Lattices Matter [4]: Concept Hierarchies and Ontologies

Example: Knowledge representation uses *concept lattices*.

For example, consider a biological taxonomy:



- Elements: Biological concepts (e.g., Animal, Mammal, Dog)
- Order: “Concept A $\leq$ Concept B” if A is a more specific type of B, e.g., “Dog $\leq$ Mammal”
- Join: Most specific common ancestor, e.g., “Mammal $\vee$ Bird $=$ Animal”
- Meet: Most general common descendant, e.g., “Bird $\wedge$ Eagle $=$ Eagle”

Why Lattices Matter [5]: Distributed Systems and Causality

Example: In distributed systems, *events* form a lattice *under causality*:

- Elements: System events with vector timestamps
- Order: "Event A $\leq$ Event B" if A causally precedes B
- Join: Latest information from both events
- Meet: Common causal history

This structure is crucial for:

- Consistent distributed databases
- Version control systems (Git DAG)
- Blockchain consensus algorithms

Why Lattices Matter [6]: Logic and Boolean Reasoning

Example: *Propositional formulas* form lattices:

- Elements: Boolean formulas over variables
- Order: $\varphi \leq \psi$ if φ implies ψ (semantic entailment)
- Join: Disjunction ($\lor$) — weaker condition
- Meet: Conjunction ($\land$) — stronger condition

Special case: *Boolean algebra* ($\texttt{true}$, $\texttt{false}$, $\lor$, $\land$, $\neg$) used in:

- Digital circuit design
- Database query languages (SQL WHERE clauses)
- Search engines (Boolean search)

TODO

- Applications of lattices in:
  - Formal concept analysis
  - Domain theory in computer science
  - Algebraic topology
  - Cryptography (lattice-based cryptography)
- Advanced topics in set theory:
  - Cardinal arithmetic
  - Ordinal numbers
  - Forcing and independence results
  - Large cardinals
- Connections to Boolean algebra (next lecture)
- Applications in formal logic and proof theory

Looking Ahead: Boolean Algebra

The next lecture will explore *Boolean algebra*, which provides the mathematical foundation for:

- Digital circuit design and computer hardware
- Propositional logic and automated reasoning
- Database query optimization
- Formal verification of software and hardware systems

Key topics will include:

- Boolean functions and their representations
- Normal forms (CNF, DNF)
- Minimization techniques (Karnaugh maps, Quine-McCluskey)
- Functional completeness and Post's theorem
- The satisfiability problem (SAT) and its computational complexity

Preview: Formal Logic

Following Boolean algebra, we will study *formal logic*, covering:

- Propositional and predicate logic
- Natural deduction and proof systems
- Completeness and soundness theorems
- Applications to program verification and AI reasoning

This progression from sets $\to$ relations $\to$ functions $\to$ Boolean algebra $\to$ logic provides a solid foundation for advanced topics in discrete mathematics and computer science.