- 1. Perform the following steps:
  - (a) Calculate the SHA-256 hash h of the string s = "DM Fall 2024 HW3" (without quotes, with all spaces, encoded in UTF-8). Convert hash h to a 256-bit binary string b (prepend leading zeros if necessary). Cut the binary string b into eight 32-bit slices  $r_1, \ldots, r_8$ , e.g.  $r_2 = b_{33..64}$ . Xor all slices into a 32-bit string  $d = r_1 \oplus \cdots \oplus r_8$ . Compute  $w = d \oplus 0$ x7613a0ca. *Hint:* last (least significant) bits of h are ...00100100, last bits of d are ...1111.
  - (b) Draw the Karnaugh map (use a template below) for a function f(A, B, C, D, E) defined by the truth table  $w = (w_1 \dots w_{32})$ , where MSB corresponds to  $f(\mathbb{O}) = w_1$  and LSB to  $f(\mathbb{I}) = w_{32}$ .
  - (c) Use K-map to find the minimal DNF and minimal CNF for the function f.
  - (d) Use K-map to find the number of prime implicants, i.e. the size of BCF.

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2. For each given function  $f_i$  of 4 arguments, draw the Karnaugh map and use it to find BCF, minimal DNF, and minimal CNF. Additionally, construct ANF (Zhegalkin polynomial) using either the K-map, the tabular ("triangle") method or the Pascal method — use each method at least once.

**Note:** WolframAlpha interprets the query "n-th Boolean function of k variables" in a reverse manner. In order to employ WolframAlpha properly, manually flip the truth table beforehand, e.g. the correct query for  $f_{10}^{(2)}$  is "5th Boolean function of 2 variables" which gives  $f_{10}^{(2)} = \neg x_2$ , since rev $(1010_2) = 0101_2 = 5_{10}$ .

(a) 
$$f_1 = f_{47541}^{(4)}$$

(c) 
$$f_3 = f_{51011}^{(4)} \oplus f_{40389}^{(4)}$$

(b) 
$$f_2 = \sum m(1, 4, 5, 6, 8, 12, 13)$$

(d) 
$$f_4 = A\overline{B}D + \overline{A}\overline{C}D + \overline{B}C\overline{D} + A\overline{C}D$$

3. Convert the following formulae to CNF.

(a) 
$$X \leftrightarrow (A \land B)$$

(d) majority 
$$(X_1, X_2, X_3)^{-1}$$

(b) 
$$Z \leftrightarrow \bigvee_i C_i$$

(e) 
$$R \to (S \to (T \to \bigwedge_i F_i))$$

(c) 
$$D_1 \oplus \cdots \oplus D_n$$

(f) 
$$M \to (H \leftrightarrow \bigvee_i D_i)$$

4. For each given system of functions  $F_i$ , determine whether it is functionally complete using Post's criterion. For each basis  $F_i$ , use it to represent the majority (A, B, C) function. Draw a combinational Boolean circuit for each resulting formula.

(a) 
$$F_1 = \{ \land, \lor, \neg \}$$

(c) 
$$F_3 = \{ \rightarrow, \not\rightarrow \}$$

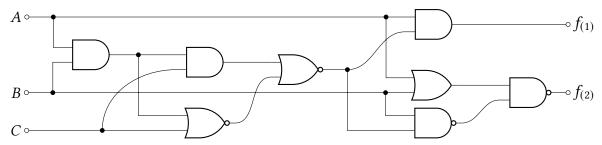
(b) 
$$F_2 = \{f_{14}^{(2)}\}$$

(d) 
$$F_4 = \{1, \leftrightarrow, \land\}$$

5. Show — without using Post's criterion — that the Zhegalkin basis  $\{\oplus, \land, 1\}$  is functionally complete.

<sup>&</sup>lt;sup>1</sup> Majority function <sup>™</sup> is a Boolean function that is 1 iff the majority (more than half) of the inputs are 1.

6. Compute the truth table for the function  $f: \mathbb{B}^3 \to \mathbb{B}^2$  (with the semantics  $\langle A, B, C \rangle \mapsto \langle f_{(1)}, f_{(2)} \rangle$ ) represented with the following circuit.



- 7. Construct a minimal Boolean circuit that implements the conversion of 4-bit binary numbers to Gray code , *i.e.* the function  $f: \mathbb{B}^4 \to \mathbb{B}^4$  with the semantics  $(b_3, b_2, b_1, b_0) \mapsto (g_3, g_2, g_1, g_0)$ , e.g.,  $0000_2 \mapsto 0000_{\text{Gray}}$ , and  $1001_2 \mapsto 1101_{\text{Gray}}$ . Use only NAND and NOR logic gates.
- 8. A *half subtractor* is a circuit that has two bits as input and produces as output a difference bit and a borrow. A *full subtractor* is a circuit that has two bits and a borrow as input, and produces as output a difference bit and a borrow.
  - (a) Construct a circuit for a half subtractor using AND gates, OR gates, and inverters.
  - (b) Construct a circuit for a full subtractor using half subtractors and NAND gates.
  - (c) Construct a circuit that computes the *saturating* difference of two four-bit integers  $(x_3x_2x_1x_0)_2$  and  $(y_3y_2y_1y_0)_2$  using half/full subtractors, AND gates, OR gates, and inverters. When  $x \ge y$ , the output bits  $d_3, \ldots, d_0$  should represent d = x y, and when x < y, the output must be zero.
- 9. Construct a circuit that compares the two-bit integers  $(x_1x_0)_2$  and  $(y_1y_0)_2$ , and outputs 1 when x > y and 0 otherwise.
- 10. Construct a circuit that computes the product of the two-bit integers  $(x_1x_0)_2$  and  $(y_1y_0)_2$ . The circuit should have four-bit output  $(p_3p_2p_1p_0)_2$  representing the product  $p = x \cdot y$ .
- 11. Consider a Boolean function ITE:  $\mathbb{B}^3 \to \mathbb{B}$  defined as follows: ITE $(c, x, y) = \begin{cases} x & \text{if } c = 0 \\ y & \text{if } c = 1 \end{cases}$ . Construct a formula for it using the standard Boolean basis  $\{\land, \lor, \neg\}$ . Determine whether the set  $\{\text{ITE}\}$  is functionally complete.
- 12. For each given function  $f_i$ , construct a Reduced Ordered Binary Decision Diagram (ROBDD) using the natural variable order  $x_1 \prec x_2 \prec \cdots \prec x_n$ . Determine whether the ROBDD can be reduced even further by using a different variable order—if so, show it.

Binary Decision Diagram (BDD) is a representation of a Boolean function as a directed acyclic graph, which consists of *decision* nodes and two *terminal* nodes (0 and 1). Each decision node is labeled by a Boolean variable  $x_i$  and has two child nodes called *low* and *high*. The edge from node to a low (high) child represents an assignment of the value FALSE (TRUE) to variable  $x_i$ . A path from the root node to the 1-terminal (0-terminal) corresponds to an assignment for which the represented Boolean function is true (false).

BDD is called *ordered* if variables appear in the same order on all paths from the root. BDD is called *reduced* if it does not contain a node v with high(v) = low(v), and there does not exist a pair of nodes u, v such that the sub-OBDDs rooted in u and v are isomorphic.

(a) 
$$f_1(x_1,...,x_4) = x_1 \oplus x_2 \oplus x_3 \oplus x_4$$

(c) 
$$f_3(x_1,...,x_4) = \sum m(1,2,5,12,15)$$

(b) 
$$f_2(x_1,...,x_5) = \text{majority}(x_1,...,x_5)$$

(d) 
$$f_4(x_1, ..., x_6) = x_1x_4 + x_2x_5 + x_3x_6$$